

Sharp three sphere inequality for perturbations of a product of two second order elliptic operators and stability for the Cauchy problem for the anisotropic plate equation *

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Abstract

We prove a sharp three sphere inequality for solutions to third order perturbations of a product of two second order elliptic operators with real coefficients. Then we derive various kinds of quantitative estimates of unique continuation for the anisotropic plate equation. Among these, we prove a stability estimate for the Cauchy problem for such an equation and we illustrate some applications to the size estimates of an unknown inclusion made of different material that might be present in the plate. The paper is self-contained and the Carleman estimate, from which the sharp three sphere inequality is derived, is proved in an elementary and direct way based on standard integration by parts.

1 Introduction

In the present paper we shall prove some quantitative estimates of unique continuation for fourth order elliptic equations arising in linear elasticity

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theory.

The equations we are most concerned with are those describing the equilibrium of a thin plate having uniform thickness. Working in the framework of the linear elasticity for infinitesimal deformations and under the kinematical assumptions of the Kirchhoff-Love theory (see [Fi], [Gu]), the transversal displacement u of the plate satisfies the following equation

$$(1.1) \quad \mathcal{L}u := \sum_{i,j,k,l=1}^2 \partial_{ij}^2 (C_{ijkl}(x) \partial_{kl}^2 u) = 0, \quad \text{in } \Omega,$$

where Ω is the middle surface of the plate and $\{C_{ijkl}(x)\}_{i,j,k,l=1}^2$ is a fourth order tensor describing the response of the material of the plate. In the sequel we shall assume that the following standard symmetry conditions are satisfied

$$(1.2) \quad C_{ijkl}(x) = C_{klij}(x) = C_{lki j}(x), \quad i, j, k, l = 1, 2, \quad \text{in } \Omega.$$

In addition we shall assume that $C_{ijkl} \in C^{1,1}(\overline{\Omega})$, $i, j, k, l = 1, 2$, and that the following strong convexity condition is satisfied

$$(1.3) \quad C_{ijkl}(x) A_{ij} A_{kl} \geq \gamma |A|^2, \quad \text{in } \Omega,$$

for every 2×2 symmetric matrix $A = \{A_{ij}\}_{i,j=1}^2$, where γ is a positive constant and $|A|^2 = \sum_{i,j=1}^2 A_{ij}^2$.

More precisely, the quantitative estimates of unique continuation which we obtain are in the form of a three sphere inequality (see Theorem 6.2, Theorem 6.5 and Theorem 6.6), in developing which we have mainly had in mind its applications to two kinds of inverse problems for thin elastic plates:

- a) the stability issue for the inverse problem of the determination of unknown boundaries,
- b) the derivation of size estimates for unknown inclusions made of different elastic material.

Let us give a brief description of problems a) and b).

Problem a). We consider a thin elastic plate, having middle surface Ω , whose boundary is made by an accessible portion Γ and by an unknown inaccessible portion I , to be determined. Assuming that the boundary portion I is free, a possible approach to determine I consists in applying a couple field \widehat{M} on Γ and measuring the resulting transversal displacement u and its normal derivative $\frac{\partial u}{\partial n}$ on an open subset of Γ . In [M-Ro] it was proved that,

under suitable a priori assumptions, a single measurement of this kind is sufficient to detect I . The stability issue, which we address here, asks whether small perturbations of the measurements produce or not small perturbations of the unknown boundary I . Since assigning a couple field \widehat{M} results in prescribing the so called Neumann conditions for the plate, that is two boundary conditions of second and third order respectively, it follows that Cauchy data are known in Γ . Therefore it is quite reasonable, also in view of the literature about stability results for the determination of unknown boundaries in other physical frameworks (see for instance [Al-B-Ro-Ve], [Si], [Ve]), that the first step to be proved in order to get such a stability result consists in stability estimates for the Cauchy problem for the fourth order equation (1.1). For this reason, in the present paper we derive a stability result for the Cauchy problem, see Theorem 3.8, having in mind applications to this inverse problem and to the analogous ones, consisting in the determination of cavities or rigid inclusions inside the plate. We refer to [M-Ro-Ve3] and to [M-Ro] respectively for uniqueness results for these two inverse problems.

Problem b). We consider a thin elastic plate, inside which an unknown inclusion made of different material might be present. Denoting by Ω and D the middle surface of the plate and of the inclusion respectively, a problem of practical interest is the evaluation of the area of D . In [M-Ro-Ve1] we derived upper and lower estimates of the area of D in terms of boundary measurements, for the case of isotropic material and assuming a “fatness” condition on the set D , see [M-Ro-Ve1, Theorem 4.1]. Since the proof of that result was mainly based on a three sphere inequality for $|\nabla^2 u|^2$ (here $\nabla^2 u$ denotes the Hessian matrix of u), where u is a solution of the plate equation, we emphasize here that Theorem 4.1 of [M-Ro-Ve1] extends to the more general anisotropic assumptions on the elasticity tensor stated in Theorem 6.5 of the present paper, in which such a three sphere inequality is established.

Concerning the Cauchy problem, along a classical path, [Ni], recently revived in [Al-R-Ro-Ve] in the framework of second order elliptic equations, we derive the stability estimates for the Cauchy problem for equation (1.1) as a consequence of smallness propagation estimates from an open set for solution to (1.1). Such smallness propagation estimates are achieved by a standard iterative application of the three sphere inequality.

In view of the applications to problems a) and b), we took care to study with particular attention the sharp character of the exponents appearing in the three sphere inequality because of its natural connection with the unique continuation property for functions vanishing at a point with polynomial rate of convergence (strong unique continuation property, [Co-Gr],

[Co-Gr-Ta], [Ge], [LeB], [L-N-W], [M-Ro-Ve1]) or with exponential rate of convergence, [Co-Ko], [Pr]. As a byproduct of our three sphere inequality, we reobtain the result in [Co-Ko], in the case of $C^{1,1}$ coefficients, stating that, if $u(x) = O\left(e^{-|x-x_0|^{-\beta}}\right)$ as $x \rightarrow x_0$, for some $x_0 \in \Omega$ and for an appropriate $\beta > 0$ which is precisely defined below, then $u \equiv 0$ in Ω . Indeed it is not worthless to stress that such kinds of unique continuation properties, especially the quantitative version of the strong unique continuation property (three sphere inequalities with optimal exponent and doubling inequalities, in the interior and at the boundary) have provided crucial tools to prove optimal stability estimates for inverse problems with unknown boundaries [Al-B-Ro-Ve], [Si], [Ve] and to get size estimates for unknown inclusions, [Al-M-Ro1], [Al-M-Ro2], [Al-M-Ro3], [Al-Ro-S], [M-Ro-Ve1], [M-Ro-Ve2]. Concerning problem b), we stress that the application of doubling inequalities allows to get size estimates of the unknown inclusion D under fully general hypotheses on D , which is assumed to be merely a measurable set, see [M-Ro-Ve2].

The strong unique continuation property for equation (1.1) holds true, [Co-Gr], [LeB], [L-N-W], [M-Ro-Ve1]), when the tensor $\{C_{ijkl}(x)\}_{i,j,k,l=1}^2$ satisfies isotropy hypotheses, that is

$$(1.4) \quad C_{ijkl}(x) = \delta_{ij}\delta_{kl}\lambda(x) + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\mu(x), \quad i, j, k, l = 1, 2, \quad \text{in } \Omega,$$

where λ and μ are the Lamé moduli.

On the other hand, in view of Alinhac Theorem [Ali], it seems extremely improbable that the solutions to (1.1) can satisfy the strong unique continuation property under the general hypotheses (1.2) and (1.3) on the tensor $\{C_{ijkl}(x)\}_{i,j,k,l=1}^2$. Indeed, let $\tilde{\mathcal{L}} = \sum_{h=0}^4 a_{4-h}(x)\partial_1^h\partial_2^{4-h}$ be the principal part of the operator \mathcal{L} . Let $z_1, z_2, \bar{z}_1, \bar{z}_2$ (here \bar{z}_j is the conjugate of the complex number z_j) be the complex roots of the algebraic equation $\sum_{h=0}^4 a_{4-h}(x_0)z^h = 0$. In [Ali] it is proved that if $z_1 \neq z_2$ then there exists an operator Q of order less than four such that the strong unique continuation property in x_0 doesn't hold true for the solutions to the equation $\tilde{\mathcal{L}}u + Qu = 0$. A fortiori, it seems hopeless the possibility that solutions to (1.1) can satisfy the doubling inequality.

At the best of our knowledge, concerning both weak and strong unique continuation property for equation (1.1), under the general assumptions (1.2), (1.3) and some reasonable smoothing condition on the coefficients C_{ijkl} , neither positive answers nor counterexamples are available in the literature. On the other hand, it is clear that, in order to face the issue of unique continuation property for equation (1.1) under the above mentioned conditions, the two-dimensional character of equation (1.1) or the specific struc-

ture of the equation should play a crucial role. Indeed, a Plis's example, [Pl], [Zu], shows that the unique continuation property fails for general three-dimensional fourth order elliptic equations with real C^∞ coefficients.

For the reasons we have just outlined, in the present paper we have a bit departed from the specific equation (1.1) and we have derived the three sphere inequality that we are interested in, as a consequence of a three sphere inequality for solutions to the equation

$$(1.5) \quad P_4(u) + Q(u) = 0, \quad \text{in } B_1 = \{x \in \mathbb{R}^n \mid |x| < 1\},$$

where $n \geq 2$, Q is a third order operator with bounded coefficients and P_4 is a fourth order elliptic operator such that

$$(1.6) \quad P_4 = L_2 L_1,$$

where L_1 and L_2 are two second order uniformly elliptic operator with real and $C^{1,1}(\overline{B_1})$ coefficients. Our approach is also supported by the fact that the operator \mathcal{L} can be written, under very general and simple conditions (see sections 3 and 6), as follows

$$(1.7) \quad \mathcal{L} = P_4 + Q,$$

where P_4 satisfies (1.6) and Q is a third order operator with bounded coefficients. We have conventionally labeled such conditions (see Definition 3.1 in Section 3) the *dichotomy condition*. On the other hand, the conditions under which the decomposition (1.7) is possible are, up to now, basically the same under which the unique continuation property holds for fourth order elliptic equation in two variables [Wat], [Zu]. More precisely, such conditions guarantee the weak unique continuation property for solution to $\mathcal{L}u = 0$ provided that the complex characteristic lines of the principal part of operator \mathcal{L} satisfy some regularity hypothesis.

We prove the three sphere inequality for solutions to equation (1.5) (provided that P_4 satisfies (1.6)) in Theorem 5.3. By such a theorem we immediately deduce, Corollary 5.4, the following unique continuation property. Let $L_k = \sum_{i,j=1}^n g_k^{ij}(x) \partial_{ij}^2$, $k = 1, 2$, where $g_k = \{g_k^{ij}(x)\}_{i,j=1}^n$ are symmetric valued function whose entries belong to $C^{1,1}(\overline{B_1})$. Assuming that $\{g_k^{ij}(x)\}_{i,j=1}^n$, $k = 1, 2$ satisfy a uniform ellipticity condition in B_1 , let ν_* and ν^* (μ_* and μ^*) be the minimum and the maximum eigenvalues of $\{g_1^{ij}(0)\}_{i,j=1}^n$ ($\{g_2^{ij}(0)\}_{i,j=1}^n$) respectively, and let $\beta > \sqrt{\frac{\mu^* \nu^*}{\mu_* \nu_*}} - 1$. We have that

$$(1.8) \quad \text{if} \quad u(x) = O\left(e^{-|x|^{-\beta}}\right), \quad \text{as } x \rightarrow 0, \quad \text{then } u \equiv 0 \quad \text{in } B_1.$$

Since (1.8) has been proved for the first time in [Co-Ko], see also [Co-Gr-Ta], where the sharp character of property (1.8) has been emphasized, we believe useful to compare our procedure with the one followed in [Co-Ko]. In the present paper, as well as in [Co-Ko], the bulk of the proof consists in obtaining a Carleman estimate for $P_4 = L_2 L_1$ with weight function $e^{-(\sigma_0(x))^{-\beta}}$, where $\beta > \sqrt{\frac{\mu^* \nu^*}{\mu_* \nu_*}} - 1$ and $(\sigma_0(x))^2$ is a suitable positive definite quadratic form (Theorem 5.2). In turn, here and in [Co-Ko], the Carleman estimate for P_4 is obtained by an iteration of two Carleman estimates for the operators L_1 and L_2 with the same weight function $e^{-(\sigma_0(x))^{-\beta}}$. However, while in [Co-Ko] and [Co-Gr-Ta] the proof of Carleman estimates for L_1 and L_2 is carried out by a careful analysis of the pseudoconvexity conditions, [Hö1], [Hö2], [Is], in the present paper, Section 4, we obtain the same estimates by a more elementary and direct way. More precisely, we adapt appropriately a technique introduced in [Es-Ve] in the context of parabolic operators. A prototype of this technique was already used in [Ke-Wa] in the issue of the boundary unique continuation for harmonic functions. Such a technique, which is based only on integration by parts and on the fundamental theorem of calculus, being direct and elementary, makes it possible to easily control the constants that occur in the final three sphere inequality.

Finally, let us notice that the above results can be extended also to treat fourth order operators having leading part $\mathcal{L}u$ given by (1.1) and involving lower order terms. An example of practical relevance is, for instance, the equilibrium problem for a thin plate resting on an elastic foundation. According to the Winkler model [Win], the corresponding equation is

$$(1.9) \quad \mathcal{L}u + ku = 0, \quad \text{in } \Omega,$$

where $k = k(x)$ is a smooth, strictly positive function. Indeed, in view of Theorem 5.3, the three sphere inequalities established in Section 6 extend to equation (1.9).

The plan of the paper is as follows. In Section 2 we introduce some basic notation. In Section 3 we present the main results for the Cauchy problem, see Theorem 3.8. In Section 4 we prove a Carleman estimate for second order elliptic operators, Theorem 4.5, which will be used in Section 5 to derive a Carleman estimate for fourth order operators obtained as composition of two second order elliptic operators, Theorem 5.2. In the same Section, as a consequence of Theorem 5.2, we also derive a three sphere inequality and the unique continuation property for such fourth order operators, see Theorem 5.3 and Corollary 5.4 respectively. Finally, in Section 6, the results of Section 5 are applied to the anisotropic plate operator, obtaining the desired three sphere inequality, see Theorems 6.2, 6.5 and 6.6.

2 Notation

Let $P = (x_1(P), x_2(P))$ be a point of \mathbb{R}^2 . We shall denote by $B_r(P)$ the ball in \mathbb{R}^2 of radius r and center P and by $R_{a,b}(P)$ the rectangle of center P and sides parallel to the coordinate axes, of length a and b , namely $R_{a,b}(P) = \{x = (x_1, x_2) \mid |x_1 - x_1(P)| < a, |x_2 - x_2(P)| < b\}$. To simplify the notation, we shall denote $B_r = B_r(O)$, $R_{a,b} = R_{a,b}(O)$.

When representing locally a boundary as a graph, we use the following definition.

Definition 2.1. ($C^{k,\alpha}$ regularity) Let Ω be a bounded domain in \mathbb{R}^2 . Given k, α , with $k \in \mathbb{N}$, $0 < \alpha \leq 1$, we say that a portion S of $\partial\Omega$ is of *class $C^{k,\alpha}$ with constants $\rho_0, M_0 > 0$* , if, for any $P \in S$, there exists a rigid transformation of coordinates under which we have $P = 0$ and

$$\Omega \cap R_{\frac{\rho_0}{M_0}, \rho_0} = \{x = (x_1, x_2) \in R_{\frac{\rho_0}{M_0}, \rho_0} \mid x_2 > \psi(x_1)\},$$

where ψ is a $C^{k,\alpha}$ function on $\left(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0}\right)$ satisfying

$$\psi(0) = 0,$$

$$\psi'(0) = 0, \quad \text{when } k \geq 1,$$

$$\|\psi\|_{C^{k,\alpha}\left(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0}\right)} \leq M_0 \rho_0.$$

When $k = 0$, $\alpha = 1$, we also say that S is of *Lipschitz class with constants ρ_0, M_0* .

Remark 2.2. We use the convention to normalize all norms in such a way that their terms are dimensionally homogeneous with the L^∞ norm and coincide with the standard definition when the dimensional parameter equals one. For instance, the norm appearing above is meant as follows

$$\|\psi\|_{C^{k,\alpha}\left(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0}\right)} = \sum_{i=0}^k \rho_0^i \|\psi^{(i)}\|_{L^\infty\left(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0}\right)} + \rho_0^{k+\alpha} |\psi^{(k)}|_\alpha\left(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0}\right),$$

where

$$|\psi^{(k)}|_\alpha\left(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0}\right) = \sup_{\substack{x', y' \in \left(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0}\right) \\ x' \neq y'}} \frac{|\psi^{(k)}(x') - \psi^{(k)}(y')|}{|x' - y'|^\alpha}.$$

Similarly, denoting by $\nabla^i u$ the vector which components are the derivatives of order i of the function u ,

$$\|u\|_{C^{k,1}(\Omega)} = \sum_{i=0}^{k+1} \rho_0^i \|\nabla^i u\|_{L^\infty(\Omega)},$$

$$\|u\|_{L^2(\Omega)} = \rho_0^{-1} \left(\int_{\Omega} u^2 \right)^{\frac{1}{2}},$$

$$\|u\|_{H^m(\Omega)} = \rho_0^{-1} \left(\sum_{i=0}^m \rho_0^{2i} \int_{\Omega} |\nabla^i u|^2 \right)^{\frac{1}{2}},$$

and so on for boundary and trace norms such as $\|\cdot\|_{H^{\frac{1}{2}}(\partial\Omega)}$, $\|\cdot\|_{H^{-\frac{1}{2}}(\partial\Omega)}$.

Notice also that, when $\Omega = B_R(0)$, then Ω satisfies Definition 2.1 with $\rho_0 = R$, $M_0 = 2$ and therefore, for instance,

$$\|u\|_{H^m(B_R)} = R^{-1} \left(\sum_{i=0}^m R^{2i} \int_{B_R} |\nabla^i u|^2 \right)^{\frac{1}{2}},$$

Given a bounded domain Ω in \mathbb{R}^2 such that $\partial\Omega$ is of class $C^{k,\alpha}$, with $k \geq 1$, we consider as positive the orientation of the boundary induced by the outer unit normal n in the following sense. Given a point $P \in \partial\Omega$, let us denote by $\tau = \tau(P)$ the unit tangent at the boundary in P obtained by applying to n a counterclockwise rotation of angle $\frac{\pi}{2}$, that is

$$(2.1) \quad \tau = e_3 \times n,$$

where \times denotes the vector product in \mathbb{R}^3 , $\{e_1, e_2\}$ is the canonical basis in \mathbb{R}^2 and $e_3 = e_1 \times e_2$.

Given any connected component \mathcal{C} of $\partial\Omega$ and fixed a point $P \in \mathcal{C}$, let us define as positive the orientation of \mathcal{C} associated to an arclength parametrization $\varphi(s) = (x_1(s), x_2(s))$, $s \in [0, l(\mathcal{C})]$, such that $\varphi(0) = P$ and $\varphi'(s) = \tau(\varphi(s))$. Here $l(\mathcal{C})$ denotes the length of \mathcal{C} .

Throughout the paper, we denote by $\partial_i u$, $\partial_s u$, and $\partial_n u$ the derivatives of a function u with respect to the x_i variable, to the arclength s and to the normal direction n , respectively, and similarly for higher order derivatives.

We denote by \mathbb{M}^2 the space of 2×2 real valued matrices and by $\mathcal{L}(X, Y)$ the space of bounded linear operators between Banach spaces X and Y .

For every 2×2 matrices A , B and for every $\mathbb{L} \in \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2)$, we use the following notation:

$$(2.2) \quad (\mathbb{L}A)_{ij} = L_{ijkl}A_{kl},$$

$$(2.3) \quad A \cdot B = A_{ij}B_{ij},$$

$$(2.4) \quad |A| = (A \cdot A)^{\frac{1}{2}},$$

$$(2.5) \quad A^{sym} = \frac{1}{2} (A + A^t),$$

where A^t denotes the transpose of the matrix A . Notice that here and in the sequel summation over repeated indexes is implied.

3 Stability estimates for the Cauchy problem

Let us consider a thin plate $\Omega \times [-\frac{h}{2}, \frac{h}{2}]$ with middle surface represented by a bounded domain Ω in \mathbb{R}^2 and having uniform thickness h , $h \ll \text{diam}(\Omega)$. Given a positive constant M_1 , we assume that

$$(3.1) \quad |\Omega| \leq M_1 \rho_0^2.$$

Let us assume that the plate is made of nonhomogeneous linear elastic material with elasticity tensor $\mathbb{C}(x) \in \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2)$ and that body forces inside Ω are absent. We denote by \hat{M} a couple field acting on the boundary $\partial\Omega$.

We shall assume throughout that the elasticity tensor \mathbb{C} has cartesian components C_{ijkl} which satisfy the following conditions

$$(3.2) \quad C_{ijkl} = C_{klij} = C_{klji} \quad i, j, k, l = 1, 2, \text{ a.e. in } \Omega.$$

We recall that the symmetry conditions (3.2) are equivalent to

$$(3.3) \quad \mathbb{C}A = \mathbb{C}A^{sym},$$

$$(3.4) \quad \mathbb{C}A \text{ is symmetric},$$

$$(3.5) \quad \mathbb{C}A \cdot B = \mathbb{C}B \cdot A,$$

for every 2×2 matrices A, B .

In order to simplify the presentation, we shall assume that the tensor \mathbb{C} is defined in all of \mathbb{R}^2 .

On the elasticity tensor \mathbb{C} we make the following assumptions:

I) *Regularity*

$$(3.6) \quad \mathbb{C} \in C^{1,1}(\mathbb{R}^2, \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2)),$$

with

$$(3.7) \quad \sum_{i,j,k,l=1}^2 \sum_{m=0}^2 \rho_0^m \|\nabla^m C_{ijkl}\|_{L^\infty(\mathbb{R}^2)} \leq M,$$

where M is a positive constant;

II) *Ellipticity (strong convexity)* There exists $\gamma > 0$ such that

$$(3.8) \quad \mathbb{C}A \cdot A \geq \gamma |A|^2, \quad \text{in } \mathbb{R}^2,$$

for every 2×2 symmetric matrix A .

Condition (3.2) implies that instead of 16 coefficients we actually deal with 6 coefficients and we denote

$$(3.9) \quad \begin{cases} C_{1111} = A_0, & C_{1122} = C_{2211} = B_0, \\ C_{1112} = C_{1121} = C_{1211} = C_{2111} = C_0, \\ C_{2212} = C_{2221} = C_{1222} = C_{2122} = D_0, \\ C_{1212} = C_{1221} = C_{2112} = C_{2121} = E_0, \\ C_{2222} = F_0, \end{cases}$$

and

$$(3.10) \quad a_0 = A_0, \quad a_1 = 4C_0, \quad a_2 = 2B_0 + 4E_0, \quad a_3 = 4D_0, \quad a_4 = F_0.$$

Let $S(x)$ be the following 7×7 matrix

$$(3.11) \quad S(x) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ 4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 & 0 \\ 0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 \\ 0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 \\ 0 & 0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 \end{pmatrix},$$

and

$$(3.12) \quad \mathcal{D}(x) = \frac{1}{a_0} |\det S(x)|.$$

Let us introduce the fourth order *plate tensor*

$$(3.13) \quad \mathbb{P} = \frac{h^3}{12} \mathbb{C}, \quad \text{in } \mathbb{R}^2.$$

With this notation we may rewrite the plate equation (1.1) in the equivalent compact form

$$(3.14) \quad \operatorname{div}(\operatorname{div}(\mathbb{P}\nabla^2 u)) = 0, \quad \text{in } \Omega,$$

where the divergence of a second order tensor field $T(x)$ is defined, as usual, by

$$(\operatorname{div} T(x))_i = \partial_j T_{ij}(x).$$

Our approach to the Cauchy problem leads us to consider the following complete, inhomogeneous equation

$$(3.15) \quad \operatorname{div}(\operatorname{div}(\mathbb{P}\nabla^2 u)) = f + \operatorname{div} F + \operatorname{div}(\operatorname{div} \mathcal{F}), \quad \text{in } B_R,$$

where $f \in L^2(\mathbb{R}^2)$, $F \in L^2(\mathbb{R}^2; \mathbb{R}^2)$, $\mathcal{F} \in L^2(\mathbb{R}^2; \mathbb{M}^2)$ satisfy the bound

$$(3.16) \quad \|f\|_{L^2(\mathbb{R}^2)} + \frac{1}{\rho_0} \|F\|_{L^2(\mathbb{R}^2; \mathbb{R}^2)} + \frac{1}{\rho_0^2} \|\mathcal{F}\|_{L^2(\mathbb{R}^2; \mathbb{M}^2)} \leq \frac{\epsilon}{\rho_0^4},$$

for a given $\epsilon > 0$.

A weak solution to (3.15) is a function $u \in H^2(B_R)$ satisfying

$$(3.17) \quad \int_{B_R} \mathbb{P}\nabla^2 u \cdot \nabla^2 \varphi = \int_{B_R} f \varphi - \int_{B_R} F \cdot \nabla \varphi + \int_{B_R} \mathcal{F} \cdot \nabla^2 \varphi, \quad \text{for every } \varphi \in H_0^2(B_R).$$

In the sequel we shall use the following condition on the elasticity tensor that we have conventionally labeled *dichotomy condition*.

Definition 3.1. (Dichotomy condition) Let \mathcal{O} be an open set of \mathbb{R}^2 . We shall say that the tensor \mathbb{P} satisfies the *dichotomy condition* in \mathcal{O} if one of the following conditions holds true

$$(3.18a) \quad \mathcal{D}(x) > 0, \quad \text{for every } x \in \overline{\mathcal{O}},$$

$$(3.18b) \quad \mathcal{D}(x) = 0, \quad \text{for every } x \in \overline{\mathcal{O}},$$

where $\mathcal{D}(x)$ is defined by (3.12).

Remark 3.2. Whenever (3.18a) holds we denote

$$(3.19) \quad \delta_1 = \min_{\overline{\mathcal{O}}} \mathcal{D}.$$

We emphasize that, in all the following statements, whenever a constant is said to depend on δ_1 (among other quantities) it is understood that such dependence occurs *only* when (3.18a) holds.

Remark 3.3. Let us briefly comment the *dichotomy condition* in the special class of *orthotropic* materials, frequently used in practical applications. In particular, let us assume that through each point of the plate there pass three mutually orthogonal planes of elastic symmetry and that these planes are parallel at all points. In this case

$$(3.20) \quad C_0 = 0, \quad D_0 = 0,$$

so that

$$(3.21) \quad a_0 = A_0, \quad a_1 = 0, \quad a_2 = 2B_0 + 4E_0, \quad a_3 = 0, \quad a_4 = F_0,$$

and

$$(3.22) \quad \mathcal{D}(x) = 16a_0a_4(a_2^2 - 4a_0a_4)^2.$$

Since, by the ellipticity condition (3.8), the coefficients a_0, a_4 are strictly positive, the dichotomy condition reduces to the vanishing or not vanishing of the factor $a_2^2 - 4a_0a_4$.

Introducing the engineering constitutive coefficients $E_1, E_2, G_{12}, \nu_{12}, \nu_{21}$, with $\nu_{12}E_2 = \nu_{21}E_1$ by the symmetry of \mathbb{C} , we have

$$(3.23) \quad a_2^2 - 4a_0a_4 = 4E_1^2 \left(\left(\frac{\nu_{12}}{k} + \frac{1 - \frac{\nu_{12}^2}{k}}{m + \nu_{12}} \right)^2 - \frac{1}{k} \right),$$

where

$$(3.24) \quad k = \frac{E_1}{E_2}, \quad m = \frac{E_1}{2G_{12}} - \nu_{12}.$$

The *isotropic* case corresponds to $k = 1$ and $m = 1$, so that, by (3.23), $\mathcal{D}(x) \equiv 0$.

Let us notice that

$$(3.25) \quad \text{if } m = \sqrt{k}, \quad \text{then } \mathcal{D}(x) \equiv 0.$$

This shows that there exist anisotropic materials such that (3.18b) is satisfied. Roughly speaking, this simple example makes clear that the value of $\mathcal{D}(x)$ cannot be interpreted as a “measure of anisotropy”.

Moreover, a case of practical interest corresponds to the vanishing of the Poisson’s coefficient ν_{12} , which gives

$$(3.26) \quad a_2^2 - 4a_0a_4 = 4E_1^2 \left(\frac{1}{m^2} - \frac{1}{k} \right),$$

so that

$$(3.27) \quad \text{if } m \neq \sqrt{k}, \quad \text{then } \mathcal{D}(x) > 0.$$

This gives an explicit class of examples in which (3.18a) holds.

Theorem 3.4 (Three sphere inequality - complete equation). *Let $u \in H^4(B_R)$ be a solution to the equation (3.15), where \mathbb{P} , defined by (3.13), satisfies (3.2), (3.7), (3.8) and the dichotomy condition in B_R . There exist positive constants k and s , $k \in (0, 1)$ only depending on γ and M , $s \in (0, 1)$ only depending on γ , M and on $\delta_1 = \min_{\overline{B_R}} \mathcal{D}$, such that for every r_1, r_2, r_3 , $0 < r_1 < r_2 < kr_3 < sR$, the following inequality holds*

$$(3.28) \quad \|u\|_{L^2(B_{r_2})} \leq C \left(\|u\|_{L^2(B_{r_1})} + \epsilon \right)^\alpha \left(\|u\|_{H^4(B_{r_3})} + \epsilon \right)^{1-\alpha}$$

where $C > 0$ and $\alpha \in (0, 1)$ only depend on γ , M , δ_1 , $\frac{r_2}{r_1}$, $\frac{r_3}{r_2}$ and $\delta_1 = \min_{\overline{B_R}} \mathcal{D}$.

Proof. Let us consider the unique solution u_0 to

$$(3.29) \quad \begin{cases} \operatorname{div}(\operatorname{div}(\mathbb{P}\nabla^2 u_0)) = f + \operatorname{div}F + \operatorname{div}(\operatorname{div}\mathcal{F}), & \text{in } B_R, \\ u_0 = 0, & \text{on } \partial B_R, \\ \frac{\partial u_0}{\partial \nu} = 0, & \text{on } \partial B_R. \end{cases}$$

By using the weak formulation (3.17) with $\varphi = u_0$, by the strong convexity condition (3.8), by using the bound (3.16) on the inhomogeneous term and by Poincaré inequality in $H_0^2(B_R)$, we have

$$(3.30) \quad \|u_0\|_{L^2(B_R)} \leq \|u_0\|_{H_0^2(B_R)} \leq C\epsilon,$$

with C only depending on γ .

Noticing that $u - u_0$ satisfies the hypotheses of Theorem 6.6, we have that the thesis immediately follows. \square

Let Σ be an open connected portion of $\partial\Omega$ such that Σ is of class $C^{1,1}$ with constants ρ_0 , M_0 , and there exists a point $P_0 \in \Sigma$ such that

$$(3.31) \quad R_{\frac{\rho_0}{M_0}, \rho_0}(P_0) \cap \partial\Omega \subset \Sigma.$$

We shall consider as test function space the space $H_{co}^2(\Omega \cup \Sigma)$ consisting of the functions $\varphi \in H^2(\Omega)$ having support compactly contained in $\Omega \cup \Sigma$. We denote by $H_{co}^{\frac{3}{2}}(\Sigma)$ the class of $H^{\frac{3}{2}}(\Sigma)$ traces of functions $\varphi \in H_{co}^2(\Omega \cup \Sigma)$, and by $H_{co}^{\frac{1}{2}}(\Sigma)$ the class of $H^{\frac{1}{2}}(\Sigma)$ traces of the normal derivative $\frac{\partial \varphi}{\partial n}$ of functions

$\varphi \in H_{co}^2(\Omega \cup \Sigma)$. Moreover, for every positive integer number m , we define $H^{-\frac{m}{2}}(\Sigma)$ as the dual space to $H^{\frac{m}{2}}(\Sigma)$ based on the $L^2(\Sigma)$ dual pairing. Let $g_1 \in H^{\frac{3}{2}}(\Sigma)$, $g_2 \in H^{\frac{1}{2}}(\Sigma)$ and $\widehat{M} \in H^{-\frac{1}{2}}(\Sigma; \mathbb{R}^2)$ be such that

$$(3.32) \quad \|g_1\|_{H^{\frac{3}{2}}(\Sigma)} + \rho_0 \|g_2\|_{H^{\frac{1}{2}}(\Sigma)} + \rho_0^2 \|\widehat{M}\|_{H^{-\frac{1}{2}}(\Sigma; \mathbb{R}^2)} \leq \eta,$$

for some positive constant η .

We consider the following Cauchy problem

$$(3.33) \quad \begin{cases} \operatorname{div}(\operatorname{div}(\mathbb{P}\nabla^2 u)) = 0, & \text{in } \Omega, \\ u = g_1, & \text{on } \Sigma, \\ \frac{\partial u}{\partial n} = g_2, & \text{on } \Sigma, \\ (\mathbb{P}\nabla^2 u)n \cdot n = -\widehat{M}_n, & \text{on } \Sigma, \\ \operatorname{div}(\mathbb{P}\nabla^2 u) \cdot n + ((\mathbb{P}\nabla^2 u)n \cdot \tau)_s = \widehat{M}_{\tau,s}, & \text{on } \Sigma, \end{cases}$$

where $\widehat{M}_\tau = \widehat{M} \cdot n$, $\widehat{M}_n = \widehat{M} \cdot \tau$ denote respectively the twisting moment and the bending moment applied at the boundary.

A weak solution to (3.33)–(3.37) is a function $u \in H^2(\Omega)$ such that

$$(3.38) \quad \int_{\Omega} \mathbb{P}\nabla^2 u \cdot \nabla^2 \varphi = - \int_{\Sigma} \left(\widehat{M}_{\tau,s} \varphi + \widehat{M}_n \varphi_n \right), \quad \text{for every } \varphi \in H_{co}^2(\Omega \cup \Sigma),$$

with

$$(3.39) \quad u|_{\Sigma} = g_1, \quad \frac{\partial u}{\partial n}|_{\Sigma} = g_2.$$

We denote

$$(3.40) \quad R_{\frac{\rho_0}{M_0}, \rho_0}^-(P_0) = \{(x_1, x_2) \in R_{\frac{\rho_0}{M_0}, \rho_0}(P_0) \mid x_2 < \psi(x_1)\},$$

that is

$$(3.41) \quad R_{\frac{\rho_0}{M_0}, \rho_0}^-(P_0) = R_{\frac{\rho_0}{M_0}, \rho_0}(P_0) \setminus \overline{\Omega}.$$

Lemma 3.5. *Let $g_1 \in H^{\frac{3}{2}}(\Sigma)$, $g_2 \in H^{\frac{1}{2}}(\Sigma)$. Then there exists $v \in H^2(R_{\frac{\rho_0}{M_0}, \rho_0}^-(P_0))$ such that*

$$(3.42) \quad v|_{\Sigma \cap R_{\frac{\rho_0}{M_0}, \rho_0}(P_0)} = g_1,$$

$$(3.43) \quad \frac{\partial v}{\partial n}|_{\Sigma \cap R_{\frac{\rho_0}{M_0}, \rho_0}(P_0)} = g_2$$

and

$$(3.44) \quad \|v\|_{H^2(R_{\frac{\rho_0}{M_0}, \rho_0}^-(P_0))} \leq C \left(\|g_1\|_{H^{\frac{3}{2}}(\Sigma)} + \rho_0 \|g_2\|_{H^{\frac{1}{2}}(\Sigma)} \right),$$

where $C, C > 0$, only depends on M_0 .

Proof. The proof follows the lines of the proof of Lemma 6.1 of [Al-R-Ro-Ve]. \square

Let us define

$$(3.45) \quad \tilde{u} = \begin{cases} u, & \text{in } \Omega, \\ v & \text{in } R_{\frac{\rho_0}{M_0}, \rho_0}^-(P_0), \end{cases}$$

$$(3.46) \quad \Omega_1 = \Omega \cup \left(\Sigma \cap R_{\frac{\rho_0}{M_0}, \rho_0}^-(P_0) \right) \cup R_{\frac{\rho_0}{M_0}, \rho_0}^-(P_0).$$

Since u and v share the same Dirichlet data (g_1, g_2) on Σ , we have that

$$(3.47) \quad \tilde{u} \in H^2(\Omega_1).$$

Theorem 3.6. *There exist $\tilde{f} \in L^2(\Omega_1)$, $\tilde{F} \in L^2(\Omega_1; \mathbb{R}^2)$, $\mathcal{F} \in L^2(\Omega_1; \mathbb{M}^2)$ such that*

$$(3.48) \quad \|\tilde{f}\|_{L^2(\Omega_1)} + \frac{1}{\rho_0} \|\tilde{F}\|_{L^2(\Omega_1; \mathbb{R}^2)} + \frac{1}{\rho_0^2} \|\mathcal{F}\|_{L^2(\Omega_1; \mathbb{M}^2)} \leq \frac{C\eta}{\rho_0^4}$$

and \tilde{u} satisfies in the weak sense the equation

$$(3.49) \quad \operatorname{div}(\operatorname{div}(\mathbb{P}\nabla^2 \tilde{u})) = \tilde{f} + \operatorname{div} \tilde{F} + \operatorname{div}(\operatorname{div} \tilde{\mathcal{F}}), \quad \text{in } \Omega_1.$$

Here, the constant $C, C > 0$, only depends on M_0 and γ .

Proof. Let φ be an arbitrary test function in $H_0^2(\Omega_1)$. It is clear that $\varphi|_\Omega \in H_{co}^2(\Omega \cup \Sigma)$. Denoting for simplicity $R^- = R_{\frac{\rho_0}{M_0}, \rho_0}^-(P_0)$, by (3.38) we have

$$(3.50) \quad \int_{\Omega_1} \mathbb{P}\nabla^2 \tilde{u} \cdot \nabla^2 \varphi = - \int_{\Sigma} (\widehat{M}_{\tau, s} \varphi + \widehat{M}_n \varphi_{,n}) + \int_{R^-} \mathbb{P}\nabla^2 v \cdot \nabla^2 \varphi.$$

Let us define the functional $\Psi : H_0^2(\Omega_1) \rightarrow \mathbb{R}$ as

$$(3.51) \quad \Psi(\varphi) = \int_{\Sigma} (\widehat{M}_{\tau, s} \varphi + \widehat{M}_n \varphi_{,n}) = \rho_0 \left(\frac{1}{\rho_0} \int_{\Sigma} (\widehat{M}_{\tau, s} \varphi + \widehat{M}_n \varphi_{,n}) \right).$$

By standard trace embedding and by (3.32), we have

$$(3.52) \quad |\Psi(\varphi)| \leq \rho_0 \left(\|\widehat{M}_{\tau,s}\|_{H^{-\frac{3}{2}}(\Sigma)} \|\varphi\|_{H^{\frac{3}{2}}(\Sigma)} + \|\widehat{M}_n\|_{H^{-\frac{1}{2}}(\Sigma)} \|\varphi,n\|_{H^{\frac{1}{2}}(\Sigma)} \right) \leq \\ \leq C \|\widehat{M}\|_{H^{-\frac{1}{2}}(\Sigma)} \|\varphi\|_{H_0^2(\Omega_1)} \leq \frac{C\eta}{\rho_0^2} \|\varphi\|_{H_0^2(\Omega_1)},$$

where $C, C > 0$, only depends on M_0 . Therefore, $\Psi \in H^{-2}(\Omega_1)$ and

$$(3.53) \quad \|\Psi\|_{H^{-2}(\Omega_1)} \leq \frac{C\eta}{\rho_0^2}.$$

By the well-known Riesz Representation Theorem in Hilbert spaces, we can find $f \in H_0^2(\Omega_1)$ such that $\Psi(\varphi) = \langle \varphi, f \rangle_{H_0^2(\Omega_1)}$ for every $\varphi \in H_0^2(\Omega_1)$ and

$$(3.54) \quad \|\Psi\|_{H^{-2}(\Omega_1)} = \|f\|_{H_0^2(\Omega_1)}.$$

Let us set

$$(3.55) \quad f_1 = \frac{f}{\rho_0^2}, \quad F_1 = -\nabla f, \quad \mathcal{F}_1 = \rho_0^2 \nabla^2 f.$$

Then

$$(3.56) \quad \rho_0 \|f_1\|_{L^2(\Omega_1)} + \|F_1\|_{L^2(\Omega_1; \mathbb{R}^2)} + \rho_0^{-1} \|\mathcal{F}_1\|_{L^2(\Omega_1; \mathbb{M}^2)} \leq \frac{C\eta}{\rho_0^3}.$$

By (3.50)

$$(3.57) \quad \int_{\Omega_1} \mathbb{P} \nabla^2 \tilde{u} \cdot \nabla^2 \varphi = \int_{R^-} \mathbb{P} \nabla^2 v \cdot \nabla^2 \varphi - \int_{\Omega_1} f_1 \varphi + \int_{\Omega_1} F_1 \cdot \nabla \varphi - \int_{\Omega_1} \mathcal{F}_1 \cdot \nabla^2 \varphi,$$

for every $\varphi \in H_0^2(\Omega_1)$. Denoting

$$(3.58) \quad \tilde{f} = -f_1, \quad \tilde{F} = -F_1, \quad \tilde{\mathcal{F}} = \begin{cases} -\mathcal{F}_1, & \text{in } \Omega_1, \\ \mathbb{P} \nabla^2 v - \mathcal{F}_1, & \text{in } R^-, \end{cases}$$

we obtain (3.49). By (3.58), (3.55), (3.7), (3.53), (3.54), (3.44), (3.32) we obtain (3.48). \square

Theorem 3.7 (Propagation of smallness in the interior). *Let Ω be a bounded domain in \mathbb{R}^2 satisfying (3.1) and let $B_{r_0}(x_0) \subset \Omega$ be a fixed disc. Let $r, 0 < r \leq \frac{r_0}{2}$ be fixed and let $G \subset \Omega$ be a connected open set such that $\text{dist}(G, \partial\Omega) \geq r$ and $B_{\frac{r_0}{2}}(x_0) \subset G$. Let $u \in H_{loc}^2(\Omega)$ be a weak solution to the equation*

$$(3.59) \quad \text{div}(\text{div}(\mathbb{P} \nabla^2 u_0)) = f + \text{div} F + \text{div}(\text{div} \mathcal{F}), \quad \text{in } \Omega$$

where \mathbb{P} , defined by (3.13), satisfies (3.2), (3.7), (3.8) and the dichotomy condition in G . Let f , F , \mathcal{F} satisfy (3.16). Let us assume that

$$(3.60) \quad \|u\|_{L^2(B_{r_0}(x_0))} \leq \eta,$$

$$(3.61) \quad \|u\|_{L^2(\Omega)} \leq E_0,$$

for given $\eta > 0$, $E_0 > 0$. We have

$$(3.62) \quad \|u\|_{L^2(G)} \leq C(\epsilon + \eta)^\delta (E_0 + \epsilon + \eta)^{1-\delta},$$

where

$$(3.63) \quad C = C_1 \left(\frac{|\Omega|}{r^2} \right)^{\frac{1}{2}},$$

$$(3.64) \quad \delta \geq \alpha \frac{C_2 |\Omega|}{r^2},$$

with $C_1 > 0$ and α , $0 < \alpha < 1$, only depending on γ , M and δ_1 , and with C_2 only depending on γ and δ_1 , where $\delta_1 = \min_{\overline{\mathcal{D}}} \mathcal{D}$.

Proof. The proof is essentially based on an iterated application of the three sphere inequality, see [Al-R-Ro-Ve, Proof of Theorem 5.1] for details. \square

Theorem 3.8 (Local stability for the Cauchy problem). *Let $u \in H^2(\Omega)$ be a weak solution to the Cauchy problem (3.33)–(3.37), where \mathbb{P} , defined by (3.13), satisfies (3.2), (3.7), (3.8) and the dichotomy condition in the rectangle $R_{\frac{\rho_0}{M_0}, \rho_0}(P_0)$, Σ satisfies (3.31), f , F , \mathcal{F} satisfy (3.16), and g_1 , g_2 , \widehat{M} satisfy (3.32). Assuming the a priori bound*

$$(3.65) \quad \|u\|_{L^2(\Omega)} \leq E_0,$$

then

$$(3.66) \quad \|u\|_{L^2\left(R_{\frac{\rho_0}{2M_0}, \frac{\rho_0}{2}}(P_0) \cap \Omega\right)} \leq C(\epsilon + \eta)^\delta (E_0 + \epsilon + \eta)^{1-\delta},$$

where $C > 0$ and δ , $0 < \delta < 1$, only depend on γ , M , M_0 , M_1 and on $\delta_1 = \min_{\overline{\mathcal{O}}} \mathcal{D}$, where $\mathcal{O} = R_{\frac{\rho_0}{M_0}, \rho_0}(P_0)$.

Proof. Representing locally Ω in a neighborhood of P_0 as

$$\Omega \cap R_{\frac{\rho_0}{M_0}, \rho_0}(P_0) = \{(x_1, x_2) \in R_{\frac{\rho_0}{M_0}, \rho_0} \mid x_2 > \psi(x_1)\},$$

let

$$r_0 = \frac{\rho_0}{2(\sqrt{1 + M_0^2} + 1)},$$

$$x_0 = \left(0, r_0 - \frac{\rho_0}{2}\right).$$

We have that

$$B_{r_0}(x_0) \subset R_{\frac{\rho_0}{2M_0}, \frac{\rho_0}{2}}^-(P_0),$$

so that

$$\|u\|_{L^2(B_{r_0}(x_0))} \leq C\eta.$$

The thesis easily follows by applying Theorem 3.7 with $\Omega = R_{\frac{\rho_0}{M_0}, \rho_0}(P_0)$, $G = R_{\frac{\rho_0}{2M_0}, \frac{\rho_0}{2}}(P_0)$, $h = \frac{r_0}{2}$. \square

4 Carleman estimate for second order elliptic operators

In this and in the next section we consider $n \geq 2$, where n is the space dimension. Moreover, in this section we use a notation for euclidean norm and scalar product which differs from the standard one used in the other sections.

Let

$$(4.1) \quad Pu = \partial_i(g^{ij}(x)\partial_j u)$$

where $\{g^{ij}(x)\}_{i,j=1}^n$ is a symmetric matrix valued function which satisfies a uniform ellipticity condition and whose entries are Lipschitz continuous functions. In order to simplify the calculations, in the sequel we shall use some standard notations in Riemannian geometry, but always dropping the corresponding volume element in the definition of the Laplace-Beltrami metric. More precisely, denoting by $g(x) = \{g_{ij}(x)\}_{i,j=1}^n$ the inverse of the matrix $\{g^{ij}(x)\}_{i,j=1}^n$ we have $g^{-1}(x) = \{g^{ij}(x)\}_{i,j=1}^n$ and we use the following notation when considering either a smooth function v or two vector fields ξ and η

$$\text{i. } \xi \cdot \eta = \sum_{i,j=1}^n g_{ij}(x)\xi_i\eta_j, \quad |\xi|^2 = \sum_{i,j=1}^n g_{ij}(x)\xi_i\xi_j,$$

$$\begin{aligned}
& \text{ii. } \nabla v = (\partial_1 v, \dots, \partial_n v), \quad \nabla_g v(x) = g^{-1}(x) \nabla v(x), \\
& \operatorname{div}(\xi) = \sum_{i=1}^n \partial_i \xi_i, \quad \Delta_g v = \operatorname{div}(\nabla_g v), \\
& \text{iii. } (\xi, \eta)_n = \sum_{i=1}^n \xi_i \eta_i, \quad |\xi|_n^2 = \sum_{i=1}^n \xi_i^2.
\end{aligned}$$

With this notation the following formulae hold true when u , v and w are smooth functions

$$(4.2) \quad Pu = \Delta_g u, \quad \Delta_g(v^2) = 2v\Delta_g v + 2|\nabla_g v|^2$$

and

$$(4.3) \quad \int_{\mathbb{R}^n} v \Delta_g w dx = \int_{\mathbb{R}^n} w \Delta_g v dx = - \int_{\mathbb{R}^n} \nabla_g v \cdot \nabla_g w dx.$$

We shall also use the following Rellich identity

$$\begin{aligned}
(4.4) \quad & 2(B \cdot \nabla_g v) \Delta_g v = \operatorname{div} (2(B \cdot \nabla_g v) \nabla_g v - B |\nabla_g v|^2) + \\
& + (\operatorname{div} B) |\nabla_g v|^2 - 2\partial_i B^k g^{ij} \partial_j v \partial_k v + B^k \partial_k g^{ij} \partial_i v \partial_j v,
\end{aligned}$$

where $B = (B^1, \dots, B^n)$ is a smooth vector field.

We denote by $w \in C^2(\mathbb{R}^n \setminus \{0\})$ a function that we shall choose later on such that $w(x) > 0$ and $|\nabla_g w| > 0$ in $\mathbb{R}^n \setminus \{0\}$.

Given $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$, let us set

$$(4.5) \quad P_\tau(f) = w^{-\tau} P(w^\tau f),$$

$$(4.6) \quad A_w(f) = \frac{w}{|\nabla_g w|} \partial_Y f + \frac{1}{2} F_w^g f,$$

where

$$(4.7) \quad F_w^g = \frac{w \Delta_g w - |\nabla_g w|^2}{|\nabla_g w|^2},$$

$$(4.8) \quad Y = \frac{\nabla_g w}{|\nabla_g w|},$$

$$(4.9) \quad \partial_Y f = \nabla_g f \cdot Y.$$

With the notation introduced above we have

$$(4.10) \quad P_\tau(f) = P_\tau^{(s)}(f) + P_\tau^{(a)}(f),$$

where $P_\tau^{(s)}$ and $P_\tau^{(a)}$ are the symmetric and the antisymmetric part of the operator P_τ with respect to the L^2 scalar product, respectively.

More precisely we have

$$(4.11) \quad P_\tau^{(s)}(f) = \Delta_g f + \tau^2 \frac{|\nabla_g w|^2}{w^2} f$$

and

$$(4.12) \quad P_\tau^{(a)}(f) = 2\tau \frac{|\nabla_g w|^2}{w^2} A_w(f).$$

Moreover, let us denote by S_w^g the symmetric matrix $S_w^g = \{S_w^{g,ij}\}_{i,j=1}^n$, where

$$(4.13) \quad S_w^{g,ij} = \frac{1}{2} \left((\operatorname{div} B) - F_w^g \right) g^{ij} - \partial_k B^j g^{ki} - \partial_k B^i g^{kj} + B^k \partial_k g^{ij},$$

with

$$(4.14) \quad B = \frac{w}{|\nabla_g w|} Y = \frac{w \nabla_g w}{|\nabla_g w|^2}.$$

We also denote

$$(4.15) \quad \mathcal{M}_w^g = S_w^g g.$$

Notice that

$$(4.16) \quad \mathcal{M}_w^g \xi \cdot \eta = \xi \cdot \mathcal{M}_w^g \eta, \quad \text{for every } \xi, \eta \in \mathbb{R}^n$$

and, letting $\xi_g = g^{-1} \xi$, $\eta_g = g^{-1} \eta$,

$$(4.17) \quad \mathcal{M}_w^g \xi_g \cdot \eta_g = (S_w^g \xi, \eta)_n, \quad \text{for every } \xi, \eta \in \mathbb{R}^n.$$

The proof of the following lemma is straightforward.

Lemma 4.1. *Let $v \in C^2(\mathbb{R}^n \setminus \{0\})$ be a function that satisfies the conditions $v(x) > 0$, $|\nabla_g v(x)| > 0$ for every $x \in \mathbb{R}^n \setminus \{0\}$. Let S_v^g , \mathcal{M}_v^g , F_v^g and B be obtained substituting w with v in the (4.13), (4.15), (4.7) and (4.14), respectively.*

Let $\varphi \in C^2(0, +\infty)$ be such that $\varphi(s) > 0$, $\varphi'(s) > 0$, for every $s \in (0, +\infty)$. Let us denote

$$(4.18) \quad \Phi(s) = \frac{\varphi(s)}{s\varphi'(s)}.$$

We have

$$(4.19) \quad \mathcal{M}_v^g \nabla_g v = S_v^g \nabla v = 0,$$

$$(4.20) \quad F_{\varphi(v)}^g = \Phi(v)F_v^g - \Phi'(v)v,$$

$$(4.21) \quad \mathcal{M}_{\varphi(v)}^g \xi \cdot \eta = v\Phi'(v) \left(\xi \cdot \eta - \frac{(\nabla_g v \cdot \xi)(\nabla_g v \cdot \eta)}{|\nabla_g v|^2} \right) + \Phi(v)\mathcal{M}_v^g \xi \cdot \eta.$$

In the sequel we shall use the following notation

$$(4.22) \quad \nabla_g^N f = (\nabla_g v \cdot \nabla_g f) \frac{\nabla_g v}{|\nabla_g v|^2} = (\partial_Y f \cdot Y)Y,$$

$$(4.23) \quad \nabla_g^T f = \nabla_g f - \nabla_g^N f,$$

Notice that $\nabla_g^N f$ and $\nabla_g^T f$ are the normal component and the tangential component (with respect to the Riemannian metric $\{g_{ij}\}_{i,j=1}^n$) of $\nabla_g f$ to the level surface of w respectively. In particular $\nabla_g^N f$ and $\nabla_g^T f$ are invariant with respect to transformations of the type $\tilde{w} = \varphi(w)$, where φ satisfies the hypotheses of Lemma 4.1. We have

$$(4.24) \quad \nabla_g^T f \cdot Y = 0, \quad \nabla_g f = \nabla_g^N f + \nabla_g^T f,$$

$$(4.25) \quad |\nabla_g f|^2 = |\nabla_g^N f|^2 + |\nabla_g^T f|^2 = (\partial_Y f)^2 + |\nabla_g^T f|^2,$$

$$(4.26) \quad \nabla_g^N f \cdot \nabla_g^T f = 0.$$

In addition, observe that by (4.16) and (4.19) we have

$$(4.27) \quad \mathcal{M}_w^g \nabla_g f \cdot \nabla_g f = \mathcal{M}_w^g \nabla_g^T f \cdot \nabla_g^T f.$$

Lemma 4.2. *Let $w \in C^2(\mathbb{R}^n \setminus \{0\})$ be such that $w(x) > 0$, $|\nabla_g w(x)| > 0$ for every $x \in \mathbb{R}^n \setminus \{0\}$. For every $\tau \neq 0$ we have*

$$(4.28) \quad \frac{w^2}{|\nabla_g w|^2} (P_\tau(f))^2 = \frac{w^2}{|\nabla_g w|^2} (P_\tau^{(s)}(f))^2 + 4\tau^2 (\partial_Y f)^2 (1 + (2\tau)^{-1} F_w^g) + \\ + 4\tau \left(\mathcal{M}_w^g \nabla_g^T f \cdot \nabla_g^T f + \frac{1}{2} F_w^g |\nabla_g^T f|^2 \right) - \\ - 2\tau^3 \frac{|\nabla_g w|^2}{w^2} F_w^g (1 + (2\tau)^{-1} F_w^g) f^2 + 2\tau F_w^g f P_\tau(f) + \operatorname{div}(q),$$

where

$$(4.29) \quad q = \frac{2\tau w}{|\nabla_g w|} \left(2(\partial_Y f) \nabla_g f - |\nabla_g f|^2 Y + \tau^2 f^2 \frac{|\nabla_g w|^2}{w^2} Y \right).$$

Proof. By (4.10) we have

$$(4.30) \quad \frac{w^2}{|\nabla_g w|^2} (P_\tau(f))^2 = \frac{w^2}{|\nabla_g w|^2} (P_\tau^{(s)}(f))^2 + \\ + 2 \frac{w^2}{|\nabla_g w|^2} P_\tau^{(s)}(f) P_\tau^{(a)}(f) + \frac{w^2}{|\nabla_g w|^2} (P_\tau^{(a)}(f))^2.$$

Let us consider the second term at the right-hand side of (4.30). We have

$$(4.31) \quad 2 \frac{w^2}{|\nabla_g w|^2} P_\tau^{(s)}(f) P_\tau^{(a)}(f) = 4\tau \left(\Delta_g f + \tau^2 \frac{|\nabla_g w|^2}{w^2} f \right) A_w(f) = \\ = 4\tau \left(\frac{w \nabla_g w \cdot \nabla_g f}{|\nabla_g w|^2} \right) \Delta_g f + 2\tau F_w^g f \Delta_g f + 4\tau^3 \frac{|\nabla_g w|^2}{w^2} A_w(f) f = \\ = 4\tau \left(\frac{w \nabla_g w \cdot \nabla_g f}{|\nabla_g w|^2} \right) \Delta_g f + 2\tau F_w^g f \Delta_g f + 2\tau^3 \operatorname{div} \left(\frac{\nabla_g w}{w} f^2 \right).$$

Now we transform the term $4\tau \left(\frac{w \nabla_g w \cdot \nabla_g f}{|\nabla_g w|^2} \right) \Delta_g f$ by applying the Rellich identity (4.4) with $B = \frac{w \nabla_g w}{|\nabla_g w|^2}$ and $v = f$. We obtain

$$(4.32) \quad 2 \frac{w^2}{|\nabla_g w|^2} P_\tau^{(s)}(f) P_\tau^{(a)}(f) = \\ = 4\tau \mathcal{M}_w^g \nabla_g f \cdot \nabla_g f + 2\tau F_w^g |\nabla_g f|^2 + 2\tau F_w^g f \Delta_g f + \operatorname{div}(q),$$

where q is given by (4.29).

Now we transform the third term at the right-hand side of (4.32) by using the following trivial consequence of (4.10)

$$(4.33) \quad \Delta_g f = P_\tau(f) - \tau^2 \frac{|\nabla_g w|^2}{w^2} f - 2\tau \frac{|\nabla_g w|^2}{w^2} A_w(f)$$

and we obtain

$$(4.34) \quad 2\tau F_w^g f \Delta_g f = 2\tau F_w^g f P_\tau(f) - 2\tau^3 \frac{|\nabla_g w|^2}{w^2} F_w^g \left(1 + \frac{1}{\tau} F_w^g\right) f^2 - 4\tau^2 \frac{|\nabla_g w|}{w} F_w^g f \partial_Y f.$$

Now, just spreading the square in the third term at the right-hand side of (4.30), we have

$$(4.35) \quad \frac{w^2}{|\nabla_g w|^2} (P_\tau^{(a)}(f))^2 = 4\tau^2 (\partial_Y f)^2 + \tau^2 \frac{|\nabla_g w|^2}{w^2} (F_w^g)^2 f^2 + 4\tau^2 \frac{|\nabla_g w|}{w} F_w^g f \partial_Y f,$$

so that, by (4.25), (4.27), (4.30), (4.32), (4.34) and (4.35) we obtain identity (4.28). \square

In the sequel of this section we assume that the matrix $\{g^{ij}(x)\}_{i,j=1}^n$ satisfies the following conditions

$$(4.36) \quad \lambda |\xi|_n^2 \leq \sum_{i,j=1}^n g^{ij}(x) \xi_i \xi_j \leq \lambda^{-1} |\xi|_n^2, \quad \text{for every } x \in \mathbb{R}^n, \xi \in \mathbb{R}^n$$

and

$$(4.37) \quad \sum_{i,j=1}^n |g^{ij}(x) - g^{ij}(y)| \leq \Lambda |x - y|_n, \quad \text{for every } x \in \mathbb{R}^n, y \in \mathbb{R}^n,$$

where $\lambda \in (0, 1]$ and $\Lambda > 0$. Now we introduce some additional notation that we shall use in the sequel. Let $\Gamma = \{\gamma_{ij}\}_{i,j=1}^n$ be a matrix that we shall choose later on. We assume that

$$(4.38) \quad m_* |x|_n^2 \leq (\Gamma x, x)_n \leq m^* |x|_n^2, \quad \text{for every } x \in \mathbb{R}^n,$$

where m_* and m^* are the minimum and the maximum eigenvalue of Γ respectively, and $m_* > 0$. Let us denote

$$(4.39) \quad \sigma(x) = ((\Gamma x, x)_n)^{1/2}$$

and we denote

$$(4.40) \quad S^{(0)} = S_\sigma^{g(0)},$$

where we recall that

$$(4.41) \quad S_\sigma^{g(0),ij} = \frac{1}{2} ((\operatorname{div} B_0) - F_\sigma^{g(0)})g^{ij}(0) - \partial_k B_0^j g^{ki}(0) - \partial_k B_0^i g^{kj}(0)$$

and

$$(4.42) \quad B_0 = \{B_0^i\}_{i=1}^n = \left\{ \frac{\sigma(x)g^{ij}(0)\partial_j\sigma(x)}{g^{lm}(0)\partial_l\sigma(x)\partial_m\sigma(x)} \right\}_{i=1}^n,$$

$$(4.43) \quad F_\sigma^{g(0)} = \frac{\sigma(x)g^{ij}(0)\partial_{ij}^2\sigma(x) - g^{ij}(0)\partial_i\sigma(x)\partial_j\sigma(x)}{g^{ij}(0)\partial_i\sigma(x)\partial_j\sigma(x)}.$$

Moreover, for any fixed $\xi \in \mathbb{R}^n$, $(S^{(0)}\xi, \xi)_n$ is an homogeneous function with

respect to the x variable of degree 0, hence the following number is well defined

$$(4.44) \quad \omega_0 = \sup \left\{ -(S^{(0)}\xi, \xi)_n \mid g^{ij}(0)\xi_i\xi_j = 1, \ g^{ij}(0)\partial_i\sigma(x)\xi_j = 0, \ x \in \mathbb{R}^n \setminus 0 \right\}.$$

We observe that ω_0 is a nonnegative number. More precisely we have the following proposition.

Proposition 4.3. *Let $Q = \sqrt{g(0)}\Gamma^{-1}\sqrt{g(0)}$, where $\sqrt{g(0)}$ is the positive square root of the matrix $g(0)$. Let ϱ_* and ϱ^* be the minimum and the maximum eigenvalues of the matrix Q respectively. Then the following equality holds true*

$$(4.45) \quad \omega_0 = \frac{\varrho^*}{\varrho_*} - 1.$$

Proof. In order to prove (4.45), let us denote

$$(4.46) \quad K = \Gamma g^{-1}(0)\Gamma$$

and let us notice that, with the conditions

$$(4.47) \quad (g^{-1}(0)\xi, \xi)_n = 1 \quad (g^{-1}(0)\nabla\sigma(x), \xi)_n = 0$$

and with the normalization condition

$$(4.48) \quad (Kx, x)_n = 1,$$

we have

$$(4.49) \quad -(S^{(0)}\xi, \xi)_n = (\Gamma x, x)_n ((K\Gamma^{-1}Kx, x)_n + (g^{-1}(0)\Gamma g^{-1}(0)\xi, \xi)_n) - 2.$$

Moreover, by introducing the new variables

$$(4.50) \quad \eta = \left(\sqrt{g(0)}\right)^{-1} \xi, \quad y = \left(\sqrt{g(0)}\right)^{-1} \Gamma x,$$

conditions (4.47) and (4.48) become respectively

$$(4.51) \quad |\eta|_n^2 = 1, \quad (y, \eta)_n = 0,$$

and

$$(4.52) \quad |y|_n^2 = 1$$

so that expression (4.49) is equal to

$$(4.53) \quad H(y, \eta) := (Qy, y)_n ((Q^{-1}y, y)_n + (Q^{-1}\eta, \eta)_n) - 2.$$

Thus we have

$$(4.54) \quad \omega_0 = \sup \{H(y, \eta) \mid |y|_n = 1, |\eta|_n = 1, (y, \eta)_n = 0\}.$$

Now let z_* and z^* be two linearly independent unit eigenvectors of Q such that $Qz_* = \varrho_* z_*$ and $Qz^* = \varrho^* z^*$. We have

$$(4.55) \quad H(z^*, z_*) = \frac{\varrho^*}{\varrho_*} - 1,$$

hence

$$(4.56) \quad \omega_0 \geq \frac{\varrho^*}{\varrho_*} - 1.$$

In order to complete the proof of (4.45) we need to prove that

$$(4.57) \quad \omega_0 \leq \frac{\varrho^*}{\varrho_*} - 1.$$

To this aim we recall the following Kantorovich inequality [Ka], [Mi]. Let \mathcal{A} be a $m \times m$ positive definite symmetric real matrix and let α_* , α^* be the minimum and the maximum eigenvalues of \mathcal{A} respectively, then for every $X \in \mathbb{R}^m$ we have

$$(4.58) \quad (\mathcal{A}X, X)_m (\mathcal{A}^{-1}X, X)_m \leq \frac{1}{4} \left(\sqrt{\frac{\alpha^*}{\alpha_*}} + \sqrt{\frac{\alpha_*}{\alpha^*}} \right)^2 |X|_m^4.$$

Now let $m = 2n$, $X = (y, \eta)^t$ and

$$(4.59) \quad \mathcal{A} = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix},$$

we have, for every $y, \eta \in \mathbb{R}^n$ such that $|y|_n = |\eta|_n = 1$, $(y, \eta) = 0$

$$(4.60) \quad H(y, \eta) = (\mathcal{A}X, X)_{2n} (\mathcal{A}^{-1}X, X)_{2n} - (Q\eta, \eta)_n (\mathcal{A}^{-1}X, X)_{2n} - 2.$$

By Schwarz inequality we have

$$(4.61) \quad (Q\eta, \eta)_n (\mathcal{A}^{-1}X, X)_{2n} = (Q\eta, \eta)_n (Q^{-1}y, y)_n + \\ + (Q\eta, \eta)_n (Q^{-1}\eta, \eta)_n \geq \frac{\varrho_*}{\varrho^*} + |\eta|_n^2 = \frac{\varrho_*}{\varrho^*} + 1.$$

On the other hand, the first term on the right-hand side of (4.60) can be estimated from above by inequality (4.58). By the obtained inequality and by (4.61) we get (4.57), that completes the proof of (4.45). \square

In the next Lemma and in the sequel we shall use the following notation when dealing with a matrix $A = \{a_{ij}\}_{i,j=1}^n$

$$(4.62) \quad |A| = \left(\sum_{i,j=1}^n a_{ij}^2 \right)^{1/2}.$$

Lemma 4.4. *There exists a constant $C, C \geq 1$, depending only on λ, Λ, m_* and m^* such that for every $x \in \mathbb{R}^n \setminus \{0\}$, $0 < \sigma(x) \leq 1$, the following inequalities hold true*

$$(4.63) \quad C^{-1} \leq |\nabla_g \sigma| \leq C, \quad |F_\sigma^g| \leq C, \quad |S^{(0)}| \leq C,$$

$$(4.64) \quad |F_\sigma^g - F_\sigma^{g(0)}| \leq C\sigma, \quad |S_\sigma^g - S^{(0)}| \leq C\sigma,$$

$$(4.65) \quad \mathcal{M}_w^g \nabla_g^T f \cdot \nabla_g^T f \geq -(\omega_0 + C\sigma) |\nabla_g^T f|^2.$$

Proof. The proof of (4.63) and (4.64) is straightforward. We prove inequality (4.65). Denote by

$$(4.66) \quad \zeta = g \nabla_g^T f.$$

We have by (4.36), (4.37), (4.64) and (4.66)

$$(4.67) \quad \begin{aligned} \mathcal{M}_w^g \nabla_g^T f \cdot \nabla_g^T f &= (S_\sigma^g \zeta, \zeta)_n \geq \\ &\geq (S^{(0)} \zeta, \zeta)_n - |((S_\sigma^g - S^{(0)}) \zeta, \zeta)_n| \geq (S^{(0)} \zeta, \zeta)_n - C\sigma |\nabla_g^T f|^2, \end{aligned}$$

where C depends only on λ, Λ, m_* and m^* .

Now, let us consider the term $(S_\sigma^g \zeta, \zeta)_n$ on the right-hand side of (4.67). Denoting by

$$(4.68) \quad \tilde{\zeta} = \zeta + g(0) (g^{-1}(x) - g^{-1}(0)) \zeta,$$

we have $g^{-1}(0) \tilde{\zeta} = g^{-1}(x) \zeta = \nabla_g^T f$, hence

$$(4.69) \quad g^{ij}(0) \tilde{\zeta}_j \partial_i \sigma = \nabla_g^T f \cdot \nabla_g \sigma = 0.$$

In addition we have

$$(4.70) \quad |\zeta - \tilde{\zeta}|_n \leq C |\nabla_g^T f| \sigma$$

and

$$(4.71) \quad g^{ij}(0) \tilde{\zeta}_j \tilde{\zeta}_i \leq (1 + C\sigma) |\nabla_g^T f|^2,$$

where C depends only on λ, Λ, m_* and m^* . By (4.44), (4.63), (4.69) and

(4.70), we obtain, for every $x \in \mathbb{R}^n \setminus \{0\}$ such that $0 < \sigma(x) \leq 1$,

$$\begin{aligned}
(4.72) \quad (S^{(0)}\zeta, \zeta)_n &\geq (S^{(0)}\tilde{\zeta}, \tilde{\zeta})_n - \\
&\quad - \left| (S^{(0)}(\zeta - \tilde{\zeta}), \zeta - \tilde{\zeta})_n \right| - 2 \left| (S^{(0)}(\zeta - \tilde{\zeta}), \tilde{\zeta})_n \right| \geq \\
&\geq -\omega_0(g^{-1}(0)\tilde{\zeta}, \tilde{\zeta})_n - C|\zeta - \tilde{\zeta}|_n^2 - 2C|\zeta - \tilde{\zeta}|_n|\tilde{\zeta}|_n \geq \\
&\geq -(\omega_0 + C\sigma)|\nabla_g^T f|^2,
\end{aligned}$$

where C depends only on λ, Λ, m_* and m^* . By the just obtained inequality and by (4.67) we obtain (4.65). \square

Let r be a given positive number, in the sequel we shall denote by B_r^σ the set $\{x \in \mathbb{R}^n | \sigma(x) < r\}$. In addition, in order to simplify the notation, we shall denote $\int_{\mathbb{R}^n}(\cdot)dx$ simply by \int and, instead to write “ f is a function that belongs to $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and f is such that $\text{supp}(f) \subset B_r^\sigma \setminus \{0\}$ ”, we shall write simply “ $f \in C_0^\infty(B_r^\sigma \setminus \{0\})$ ”.

Theorem 4.5. *Let β be a number such that $\beta > \omega_0$, let*

$$(4.73) \quad \varphi(s) = e^{-s^{-\beta}}$$

and let $w(x) = \varphi(\sigma(x))$. There exist constants C , τ_1 and r_0 , ($C \geq 1$, $\tau_1 \geq 1$, $0 < r_0 \leq 1$) depending only on $\lambda, \Lambda, m_, m^*$ and β such that for every $u \in C_0^\infty(B_{r_0}^\sigma \setminus \{0\})$ and for every $\tau \geq \tau_1$ the following inequality holds true*

$$(4.74) \quad \tau \int \sigma^\beta w^{-2\tau} |\nabla_g u|^2 + \tau^3 \int \sigma^{-\beta-2} w^{-2\tau} u^2 \leq C \int \sigma^{2\beta+2} w^{-2\tau} (\Delta_g u)^2.$$

Proof. Let $w(x) = \varphi(\sigma(x))$, where $\sigma(x) = ((\Gamma x, x)_n)^{1/2}$. Let us notice that φ satisfies the hypotheses of Lemma 4.1 and that

$$(4.75) \quad \Phi(s) = \frac{s^\beta}{\beta}.$$

Let $u \in C_0^\infty(B_1^\sigma \setminus \{0\})$ and $f = w^{-\tau}u$. By (4.21) and by (4.65) we have

$$(4.76) \quad \mathcal{M}_w^g \nabla_g^T f \cdot \nabla_g^T f \geq \sigma^\beta \left(1 - \frac{\omega_0}{\beta} - C\sigma \right) |\nabla_g^T f|^2,$$

where C depends only on $\lambda, \Lambda, m_*, m^*$ and β . Now, denoting

$$(4.77) \quad \psi_0 = \sigma^\beta \left(-1 + \frac{1}{\beta} F_\sigma^{g(0)} \right),$$

by (4.20) we have

$$(4.78) \quad F_w^g = \psi_0 + \frac{\sigma^\beta}{\beta} (F_\sigma^g - F_\sigma^{g(0)}),$$

hence by (4.63) and (4.64) of Lemma 4.4 we have, for every $x \in B_1^\sigma \setminus \{0\}$,

$$(4.79) \quad |F_w^g| \leq C\sigma^\beta, \quad |F_w^g - \psi_0| \leq C\sigma^{\beta+1},$$

where $C, C \geq 1$, depends only on $\lambda, \Lambda, m_*, m^*$ and β .

Let ψ_1 be a function that we shall choose later on, by (4.11) we have

$$(4.80) \quad \begin{aligned} & \frac{w^2}{|\nabla_g w|^2} (P_\tau^{(s)}(f))^2 = \\ & = \frac{w^2}{|\nabla_g w|^2} \left(P_\tau^{(s)}(f) - \tau \frac{|\nabla_g w|^2}{w^2} \psi_1 f + \tau \frac{|\nabla_g w|^2}{w^2} \psi_1 f \right)^2 \geq \\ & \geq 2\tau \psi_1 f \left(P_\tau^{(s)}(f) - \tau \frac{|\nabla_g w|^2}{w^2} \psi_1 f \right) = \\ & = 2\tau^3 \left(\left(1 - \frac{\psi_1}{\tau} \right) \psi_1 \frac{|\nabla_g w|^2}{w^2} + \frac{1}{2\tau^2} \Delta_g \psi_1 \right) f^2 - 2\tau \psi_1 |\nabla_g f|^2 + \operatorname{div}(q_1), \end{aligned}$$

where

$$(4.81) \quad q_1 = \tau (2\psi_1 f \nabla_g f - f^2 \nabla_g \psi_1).$$

By inequalities (4.76) and (4.80), by (4.25) and by Lemma 4.2 we obtain

$$(4.82) \quad \begin{aligned} & \frac{w^2}{|\nabla_g w|^2} (P_\tau(f))^2 \geq 2\tau^3 a_1 f^2 + \\ & + 4\tau a_2 |\nabla_g^T f|^2 + 4\tau^2 a_3 (\partial_Y f)^2 + 2\tau F_w^g f P_\tau(f) + \operatorname{div}(q_2), \end{aligned}$$

where

$$(4.83) \quad a_1 = \frac{|\nabla_g w|^2}{w^2} \left((\psi_1 - F_w^g) - \frac{1}{\tau} \left(\frac{1}{2} (F_w^g)^2 + \psi_1^2 \right) \right) + \frac{1}{2\tau^2} \Delta_g \psi_1,$$

$$(4.84) \quad a_2 = \sigma^\beta \left(1 - \frac{\omega_0}{\beta} - C\sigma \right) + \frac{1}{2} (F_w^g - \psi_1)$$

$$(4.85) \quad a_3 = 1 + \frac{1}{2\tau}(F_w^g - \psi_1),$$

$$(4.86) \quad q_2 = q + q_1.$$

Now we choose

$$(4.87) \quad \psi_1 = \psi_0 + \frac{\varepsilon \sigma^\beta}{\beta},$$

where $0 < \varepsilon \leq \min\{1, \beta - \omega_0\}$.

Let us notice that for every $x \in B_1^\sigma \setminus \{0\}$,

$$(4.88) \quad C^{-1} \sigma^{-2\beta-2} \leq \frac{|\nabla_g w|^2}{w^2} \leq C \sigma^{-2\beta-2},$$

$$(4.89) \quad F_w^g - \psi_1 \geq -\frac{\sigma^\beta}{\beta} (\varepsilon + C\sigma),$$

$$(4.90) \quad \psi_1 - F_w^g \geq \frac{\sigma^\beta}{\beta} (\varepsilon - C\sigma),$$

$$(4.91) \quad |\psi_1| \leq C\sigma^\beta, \quad |\Delta_g \psi_1| \leq C\sigma^{\beta-2},$$

where $C, C \geq 1$, depends only on $\lambda, \Lambda, m_*, m^*$ and β , with (4.89)–(4.91) following from (4.77)–(4.79) and (4.87). From (4.88)–(4.91) we have that, for every $x \in B_1^\sigma \setminus \{0\}$ and for every $\tau \geq 1$

$$(4.92) \quad a_1 \geq C_*^{-1} \sigma^{-\beta-2} \left(\varepsilon - C_0 \sigma - \frac{C_1}{\tau} \sigma^\beta \right),$$

where C_*, C_0, C_1 ($C_* \geq 1, C_0 \geq 1, C_1 \geq 1$) depend only on $\lambda, \Lambda, m_*, m^*$ and β . Therefore, if $0 < \sigma(x) \leq \frac{\varepsilon}{2C_0}$ and $\tau \geq \frac{4C_1}{\varepsilon}$, then we have

$$(4.93) \quad a_1 \geq \frac{\varepsilon}{4} C_*^{-1} \sigma^{-\beta-2},$$

where $C, C \geq 1$, depends only on $\lambda, \Lambda, m_*, m^*$ and β .
Concerning a_2 , we have by (4.89)

$$(4.94) \quad a_2 \geq \sigma^\beta \left(\frac{1}{2} \left(1 - \frac{\omega_0}{\beta} \right) - C_2 \sigma \right),$$

where $C_2, C_2 \geq 1$, depends only on $\lambda, \Lambda, m_*, m^*$ and β . Therefore, if $0 < \sigma(x) \leq \frac{\beta - \omega_0}{4\beta C_2}$, then we have

$$(4.95) \quad a_2 \geq \frac{1}{4} \sigma^\beta \left(1 - \frac{\omega_0}{\beta} \right),$$

Concerning a_3 , by (4.91) and (4.79) we have that there exists $C_3, C_3 \geq 1$, depending only on $\lambda, \Lambda, m_*, m^*$ and β such that if $\tau \geq C_3$ and $0 < \sigma(x) \leq 1$ then

$$(4.96) \quad a_3 \geq \frac{1}{2}.$$

Now, denote by $\tau_0 = \max\{\frac{4C_1}{\varepsilon}, C_3\}$ and $r_0 = \min\{\frac{\varepsilon}{2C_0}, \frac{\beta - \omega_0}{4\beta C_2}\}$, by (4.25), (4.82), (4.93), (4.95) and (4.96) we have

$$(4.97) \quad \frac{w^2}{|\nabla_g w|^2} (P_\tau(f))^2 \geq \tau^3 \sigma^{-\beta-2} \frac{\varepsilon}{2} C_*^{-1} f^2 + \\ + \tau \sigma^\beta \left(1 - \frac{\omega_0}{\beta} \right) |\nabla_g f|^2 + 2\tau F_w^g f P_\tau(f) + \operatorname{div}(q_2),$$

for every $x \in B_{r_0}^\sigma \setminus \{0\}$ and $\tau \geq \tau_0$.

By Young's inequality, by the first of (4.79) and by (4.89) we have

$$(4.98) \quad |2\tau F_w^g f P_\tau(f)| \leq \frac{1}{2} \frac{w^2}{|\nabla_g w|^2} (P_\tau(f))^2 + C_4 \tau^2 \sigma^{-2} f^2,$$

where $C_4, C_4 \geq 1$, depends only on $\lambda, \Lambda, m_*, m^*$ and β .

By (4.97) and (4.98) we have

$$(4.99) \quad \frac{1}{2} \frac{w^2}{|\nabla_g w|^2} (P_\tau(f))^2 \geq \tau^3 \sigma^{-\beta-2} \frac{\varepsilon}{4} C_*^{-1} f^2 + \\ + \tau \sigma^\beta \left(1 - \frac{\omega_0}{\beta} \right) |\nabla_g f|^2 + \operatorname{div}(q_2),$$

for every $x \in B_{r_0}^\sigma \setminus \{0\}$ and every $\tau \geq \tau_1 := \max\{\tau_0, \frac{4C_*C_4}{\varepsilon}\}$.

Finally, we choose $\varepsilon = \min\{1, \beta - \omega_0\}$. Recalling that $f = w^{-\tau}u$, and integrating both sides of (4.99) over $B_{r_0}^\sigma \setminus \{0\}$, we obtain (4.74). \square

Remark 4.6. It is straightforward that estimate (4.74) remains valid for operators in non-divergence form $Pu = g_{ij}\partial_{ij}^2u$. Of course, the values of the constants, and in particular of τ_1 , might be different.

5 Carleman estimate for product of two second order elliptic operators

In this section and in the sequel we return to the standard notation, that is we denote by $|\cdot|$ and by \cdot the euclidian norm and scalar product respectively.

Let $\{g_1^{ij}(x)\}_{i,j=1}^n$ and $\{g_2^{ij}(x)\}_{i,j=1}^n$ be two symmetric matrix real valued functions which satisfy conditions (4.36), (4.37) and let us assume that

$$(5.1) \quad \sum_{i,j=1}^n \|\nabla^2 g_1^{ij}\|_{L^\infty(\mathbb{R}^n)} \leq \Lambda_1, \quad \sum_{i,j=1}^n \|\nabla^2 g_2^{ij}\|_{L^\infty(\mathbb{R}^n)} \leq \Lambda_1,$$

with $\Lambda_1 > 0$. Let us denote by L_1 , L_2 and \mathcal{L} the operators

$$(5.2) \quad L_1(u) = \sum_{i,j=1}^n g_1^{ij}(x) \partial_{ij}^2 u, \quad L_2(u) = \sum_{i,j=1}^n g_2^{ij}(x) \partial_{ij}^2 u,$$

$$(5.3) \quad \mathcal{L}(u) = L_2(L_1 u).$$

In the sequel we shall need the following standard proposition which we prove for the reader's convenience.

Proposition 5.1. *Let L_1 , L_2 and \mathcal{L} be the operators defined above. Given $a \in C^1(\mathbb{R}^n \setminus \{0\})$ and $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, the following inequalities hold true:*

$$(5.4) \quad \int a^2 |\nabla^2 u|^2 \leq C \left(\int a^2 |L_k u|^2 + \int (a^2 + |\nabla a|^2) |\nabla u|^2 \right), \quad k = 1, 2,$$

$$(5.5) \quad \int a^2 |\nabla^3 u|^2 \leq C \left(\int a^2 |\mathcal{L} u| |\nabla^2 u| + \int (a^2 + |\nabla a|^2) |\nabla^2 u|^2 \right),$$

where C only depends on λ and Λ .

Proof. To simplify the notation, let us omit the index k in L_k . For a fixed $l \in \{1, \dots, n\}$ we have

$$\begin{aligned}
(5.6) \quad & \int Lu \partial_{ll}^2 u a^2 = - \int \partial_l (a^2 g^{ij} \partial_{ij}^2 u) \partial_l u = \\
& = - \int a^2 g^{ij} \partial_{ijl}^3 u \partial_l u - 2 \int a \partial_l a g^{ij} \partial_{ij}^2 u \partial_l u - \int (\partial_l g^{ij}) \partial_{ij}^2 u \partial_l u a^2 = \\
& = \int a^2 g^{ij} \partial_{il}^2 u \partial_{jl}^2 u + \int \partial_j (a^2 g^{ij}) \partial_{il}^2 u \partial_l u - 2 \int a \partial_l a g^{ij} \partial_{ij}^2 u \partial_l u - \int (\partial_l g^{ij}) \partial_{ij}^2 u \partial_l u a^2 \geq \\
& \geq \lambda \int a^2 |\nabla \partial_l u|^2 - C \int (|a| + |\nabla a|) |a| |\nabla u| |\nabla^2 u|,
\end{aligned}$$

where C only depends on λ and Λ .

Now, summing up with respect to l the above inequalities and applying the inequality $2xy \leq x^2 + y^2$, we get (5.4).

Now we prove (5.5). First we observe that, [G-T], multiplying both sides of the second equality (5.2) by $a^2 v$ and integrating by parts we easily obtain

$$(5.7) \quad \int a^2 |\nabla v|^2 \leq C \left(\int a^2 |L_2 v| |v| + \int (a^2 + |\nabla a|^2) v^2 \right),$$

where C only depends on λ and Λ .

Let us apply (5.7) to $v = L_1 u$. Noticing that, for a fixed $l \in \{1, \dots, n\}$, we have

$$(5.8) \quad |L_1(\partial_l u)| \leq |\partial_l(L_1 u)| + C |\nabla^2 u|,$$

where C only depends on Λ , we obtain

$$(5.9) \quad \int a^2 |L_1(\partial_l u)|^2 \leq C \left(\int a^2 |\mathcal{L} u| |\nabla^2 u| + \int (a^2 + |\nabla a|^2) |\nabla^2 u|^2 \right),$$

where C only depends on λ and Λ .

Finally, by applying inequality (5.4) to estimate from below the integral on the left hand side of (5.9), and summing up with respect to l , we get (5.5). \square

In order to prove the next theorem we need to use some transformation formulae for the operator \mathcal{L} which we recall now. Let $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^4 diffeomorphism. We have

$$(5.10) \quad (\mathcal{L} u)(\Psi^{-1}(y)) = (\tilde{\mathcal{L}} U)(y) + (QU)(y),$$

where $U(y) = u(\Psi^{-1}(y))$, Q is a third order operator, $\tilde{\mathcal{L}} = \tilde{L}_2 \tilde{L}_1$, $\tilde{L}_k = \sum_{i,j=1}^n \tilde{g}_k^{ij}(y) \partial_{ij}^2$, $k = 1, 2$, and $\tilde{g}_k^{-1}(\Psi(x)) = \frac{\partial \Psi}{\partial x}(x) g_k^{-1}(x) \left(\frac{\partial \Psi}{\partial x}(x) \right)^t$, namely

$$(5.11) \quad \tilde{g}_k^{ij}(\Psi(x)) = \sum_{r,s=1}^n g_k^{rs}(x) \frac{\partial \Psi_i}{\partial x_r}(x) \frac{\partial \Psi_j}{\partial x_s}(x), \quad i, j = 1, \dots, n.$$

We can find a linear map Ψ such that $\tilde{g}_1^{-1}(0)$ is the identity matrix and $\tilde{g}_2^{-1}(0)$ is a diagonal matrix. More precisely, let R_1 be the matrix of a rotation such that $R_1 g_1^{-1}(0) R_1^t = \text{diag}\{\nu_1, \dots, \nu_n\}$, where ν_i , $i = 1, \dots, n$, are the eigenvalues of $g_1^{-1}(0)$, and let $H = \text{diag}\{\frac{1}{\sqrt{\nu_1}}, \dots, \frac{1}{\sqrt{\nu_n}}\}$. We have that $H R_1 g_1^{-1}(0) R_1^t H^t$ is equal to the identity matrix. Now let R_2 be the matrix of a rotation such that $\tilde{g}_2^{-1}(0) = R_2 H R_1 g_2^{-1}(0) R_1^t H^t R_2^t$ has a diagonal form. We have that the desired map is $\Psi(x) = R_2 H R_1 x$. In addition, notice that if ν_* , ν^* are the minimum and maximum eigenvalues of $g_1^{-1}(0)$ respectively and μ_* , μ^* are the minimum and maximum eigenvalues of $g_2^{-1}(0)$ respectively, then

$$(5.12) \quad \frac{\mu_*}{\nu^*} |x|^2 \leq \tilde{g}_2^{-1}(0) x \cdot x \leq \frac{\mu^*}{\nu_*} |x|^2, \quad \text{for every } x \in \mathbb{R}^n.$$

Theorem 5.2. *Let \mathcal{L} be the operator defined by (5.3). Let ν_* and ν^* (μ_* and μ^*) be the minimum and the maximum eigenvalues of $g_1^{-1}(0)$ ($g_2^{-1}(0)$). Then there exists a symmetric matrix Γ_0 satisfying*

$$(5.13) \quad \lambda^2 |x|^2 \leq \sigma_0^2(x) := \Gamma_0 x \cdot x \leq \lambda^{-2} |x|^2,$$

and such that if $\beta > \sqrt{\frac{\mu^* \nu^*}{\mu_* \nu_*}} - 1$ and

$$(5.14) \quad w_0(x) = e^{-(\sigma_0(x))^{-\beta}}$$

the following inequality holds true:

$$(5.15) \quad \sum_{k=0}^3 \tau^{6-2k} \int \sigma_0^{-\beta-2+k(2\beta+2)} w_0^{-2\tau} |\nabla^k u|^2 dx \leq C \int \sigma_0^{5\beta+6} w_0^{-2\tau} |\mathcal{L}u|^2 dx,$$

for every $u \in C_0^\infty(B_{r_1}^{\sigma_0} \setminus \{0\})$ and for every $\tau \geq \bar{\tau}$, where r_1 , $0 < r_1 < 1$, C and $\bar{\tau}$ only depend on λ , Λ and Λ_1 .

Proof. By the comments preceding the statement of the theorem, without loosing of generality we can assume that $g_1^{ij}(0) = \delta^{ij}$ and $g_2^{-1}(0)$ is of diagonal form, say $g_2^{-1}(0) = \text{diag}\{\mu_1, \mu_2, \dots, \mu_n\}$, where $0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_n$. We denote by $\Gamma = \{\gamma_{ij}\}_{i,j=1}^n$ a symmetric matrix that we shall choose later on, and by m_* and m^* the minimum and the maximum eigenvalues of Γ

respectively, with $m_* > 0$. Let us set $\sigma(x) = (\Gamma x \cdot x)^{1/2}$. We denote by $S_k^{(0)}$, $k = 1, 2$, the matrix $S_\sigma^{g_k^{(0)}}$ introduced in (4.40). We denote by ω_0^k the numbers (compare with (4.44))

$$(5.16) \quad \omega_0^k = \sup \left\{ -(S_k^{(0)} \xi) \cdot \xi \mid g_k^{ij}(0) \xi_i \xi_j = 1, g_k^{ij}(0) \partial_i \sigma(x) \xi_j = 0, x \in \mathbb{R}^n \setminus \{0\} \right\}.$$

Let β be a positive number such that $\beta > \max\{\omega_0^1, \omega_0^2\}$ and let $V \in C_0^\infty(B_{r_0}^\sigma \setminus \{0\})$, where r_0 has been defined in Theorem 4.5. Since

$$(5.17) \quad |\Delta_{g_k} V| \leq |L_k V| + C |\nabla V|, \quad k = 1, 2,$$

where C only depends on Λ , by (4.74) we have that there exists τ_2 , only depending on λ , Λ , m_* , m^* and β such that for $k = 1, 2$, and for every $\tau \geq \tau_2$

$$(5.18) \quad \tau \int \sigma^\beta w^{-2\tau} |\nabla V|^2 + \tau^3 \int \sigma^{-\beta-2} w^{-2\tau} V^2 \leq C \int \sigma^{2\beta+2} w^{-2\tau} |L_k V|^2.$$

Now we iterate inequality (5.18). First we notice that, by a standard density property, inequality (5.18) is valid for every $V \in H_0^2(B_{r_0}^\sigma \setminus \{0\})$. Let u be an arbitrary function belonging to $C_0^\infty(B_{r_0}^\sigma \setminus \{0\})$ and let us set $v = L_1 u$. By applying inequality (5.18) to the function $V = \sigma^{\frac{3}{2}\beta+2} v$, we get

$$(5.19) \quad \begin{aligned} \tau^3 \int \sigma^{2\beta+2} w^{-2\tau} v^2 &= \tau^3 \int \sigma^{-\beta-2} w^{-2\tau} (\sigma^{\frac{3}{2}\beta+2} v)^2 \leq \\ &\leq C \int \sigma^{2\beta+2} w^{-2\tau} |L_2(\sigma^{\frac{3}{2}\beta+2} v)|^2, \end{aligned}$$

for every $\tau \geq \tau_2$.

Now observe that

$$(5.20) \quad |L_2(\sigma^{\frac{3}{2}\beta+2} v)| \leq \sigma^{\frac{3}{2}\beta+2} |L_2 v| + C \sigma^{\frac{3}{2}\beta+1} |\nabla v| + C \sigma^{\frac{3}{2}\beta} |v|,$$

where C only depends on λ , Λ , m_* , m^* and β . By using (5.20) to estimate from above the right hand side of (5.19), we have that there exists $\tau_3 \geq \tau_2$ such that, for every $\tau \geq \tau_3$,

$$(5.21) \quad \tau^3 \int \sigma^{2\beta+2} w^{-2\tau} v^2 \leq C \int \sigma^{5\beta+6} w^{-2\tau} |L_2 v|^2 + C \int \sigma^{5\beta+4} w^{-2\tau} |\nabla v|^2,$$

where C and τ_3 only depend on λ , Λ , m_* , m^* and β .

Now we estimate from above the second term in the right hand side of (5.21). To this aim we apply inequality (5.18) to the function $V = \sigma^{2\beta+2} v$ and we have

$$(5.22) \quad \tau \int \sigma^\beta w^{-2\tau} |\nabla(\sigma^{2\beta+2} v)|^2 \leq C \int \sigma^{2\beta+2} w^{-2\tau} |L_2(\sigma^{2\beta+2} v)|^2,$$

for every $\tau \geq \tau_2$.

Taking into account that

$$(5.23) \quad |L_2(\sigma^{2\beta+2}v)| \leq \sigma^{2\beta+2}|L_2v| + C\sigma^{2\beta+1}|\nabla v| + C\sigma^{2\beta}|v|,$$

and

$$(5.24) \quad |\nabla(\sigma^{2\beta+2}v)|^2 \geq \frac{1}{2}\sigma^{4\beta+4}|\nabla v|^2 - C\sigma^{4\beta+2}v^2,$$

where C only depends on $\lambda, \Lambda, m_*, m^*$ and β , we have, by (5.22),

$$(5.25) \quad \tau \int \sigma^{5\beta+4}w^{-2\tau}|\nabla v|^2 \leq C \int \sigma^{6\beta+6}w^{-2\tau}|L_2v|^2 + C\tau \int \sigma^{5\beta+2}w^{-2\tau}v^2,$$

for every $\tau \geq \tau_2$, where C only depends on $\lambda, \Lambda, m_*, m^*$ and β .

Now we use (5.25) to estimate from above the second term on the right hand side of (5.21) and we have that there exists $\tau_4 \geq \tau_3$ such that

$$(5.26) \quad \int \sigma^{2\beta+2}w^{-2\tau}v^2 \leq \frac{C}{\tau^3} \int \sigma^{5\beta+6}w^{-2\tau}|L_2v|^2,$$

for every $\tau \geq \tau_4$, where C and τ_4 only depend on $\lambda, \Lambda, m_*, m^*$ and β . Recalling that $v = L_1u$ and by using (5.18) for $V = u$ and $k = 1$, (5.26) yields

$$(5.27) \quad \tau^6 \int \sigma^{-\beta-2}w^{-2\tau}u^2 + \tau^4 \int \sigma^\beta w^{-2\tau}|\nabla u|^2 \leq C \int \sigma^{5\beta+6}w^{-2\tau}|L_2L_1u|^2,$$

for every $\tau \geq \tau_4$, where C only depends on $\lambda, \Lambda, m_*, m^*$ and β .

Now we prove that

$$(5.28) \quad \tau^2 \int \sigma^{3\beta+2}w^{-2\tau}|\nabla^2 u|^2 + \int \sigma^{5\beta+4}w^{-2\tau}|\nabla^3 u|^2 \leq C \int \sigma^{5\beta+6}w^{-2\tau}|L_2L_1u|^2,$$

for every $\tau \geq \tau_4$, where C only depends on $\lambda, \Lambda, m_*, m^*$ and β .

Concerning the term with the second order derivatives on the left hand side of (5.28), we can estimate it by using (5.4) with $a = (\sigma^{3\beta+2}w^{-2\tau})^{\frac{1}{2}}$ and $k = 1$, obtaining

$$(5.29) \quad \int \sigma^{3\beta+2}w^{-2\tau}|\nabla^2 u|^2 \leq C \int \sigma^{3\beta+2}w^{-2\tau}|L_1u|^2 + C\tau^2 \int \sigma^\beta w^{-2\tau}|\nabla u|^2,$$

where C only depends on $\lambda, \Lambda, m_*, m^*$ and β .

By using (5.18) for $V = u$ and $k = 1$ to estimate from above the second integral on the right hand side of (5.29) we get

$$(5.30) \quad \int \sigma^{3\beta+2} w^{-2\tau} |\nabla^2 u|^2 \leq C\tau \int \sigma^{2\beta+2} w^{-2\tau} |L_1 u|^2,$$

for every $\tau \geq \tau_2$, where C and τ_2 only depend on $\lambda, \Lambda, m_*, m^*$ and β .

Now, by (5.26) with $v = L_1 u$ and by (5.30), we have, for every $\tau \geq \tau_4$,

$$(5.31) \quad \tau^2 \int \sigma^{3\beta+2} w^{-2\tau} |\nabla^2 u|^2 \leq C \int \sigma^{5\beta+6} w^{-2\tau} |L_2 L_1 u|^2,$$

where C only depends on $\lambda, \Lambda, m_*, m^*$ and β .

Now we estimate from above the term with the third order derivatives on the left hand side of (5.28). By applying (5.5) with $a = (\sigma^{5\beta+4} w^{-2\tau})^{\frac{1}{2}}$, we have

$$(5.32) \quad \int \sigma^{5\beta+4} w^{-2\tau} |\nabla^3 u|^2 \leq C \int \sigma^{5\beta+4} w^{-2\tau} |L_2 L_1 u| |\nabla^2 u| + C\tau^2 \int \sigma^{3\beta+2} w^{-2\tau} |\nabla^2 u|^2,$$

where C only depends on $\lambda, \Lambda, m_*, m^*$ and β .

Noticing that

$$(5.33) \quad \begin{aligned} \sigma^{5\beta+4} |L_2 L_1 u| |\nabla^2 u| &= \left(\sigma^{\frac{3}{2}\beta+1} |\nabla^2 u| \right) \left(\sigma^{\frac{7}{2}\beta+3} |L_2 L_1 u| \right) \leq \\ &\leq \frac{1}{2} \left(\sigma^{3\beta+2} |\nabla^2 u|^2 + \sigma^{7\beta+6} |L_2 L_1 u|^2 \right), \end{aligned}$$

by (5.31) and (5.32) we obtain the desired inequality (5.28).

By (5.27) and (5.28) we have

$$(5.34) \quad \sum_{k=0}^3 \tau^{6-2k} \int \sigma^{-\beta-2+k(2\beta+2)} w^{-2\tau} |\nabla^k u|^2 \leq C \int \sigma^{5\beta+6} w^{-2\tau} |L_2 L_1 u|^2,$$

for every $\tau \geq \tau_4$, where τ_4 and C only depend on $\lambda, \Lambda, m_*, m^*$ and β , for every $u \in C_0^\infty(B_{r_0}^\sigma \setminus \{0\})$.

Now we choose $\Gamma = \Gamma_0 := \text{diag}\{\frac{1}{\sqrt{\mu_1}}, \dots, \frac{1}{\sqrt{\mu_n}}\}$, $\sigma(x) = \sigma_0(x) := (\Gamma_0 x \cdot x)^{1/2}$, $w(x) = w_0(x)$, where $w_0(x)$ is defined by (5.14). By Proposition 4.3 we have $\omega_0^1 = \omega_0^2 = \sqrt{\frac{\mu_n}{\mu_1}} - 1$, hence estimate (5.34) holds for $\beta > \sqrt{\frac{\mu_n}{\mu_1}} - 1$. Coming back to the old variables we obtain (5.15). \square

Theorem 5.3. *Let \mathcal{L} be the operator defined by (5.3). Let $\nu_*, \nu^*, \mu_*, \mu^*$ be as defined in Theorem 5.2. Let us assume that $u \in H^4(B_R)$ satisfies the inequality*

$$(5.35) \quad |\mathcal{L}u| \leq N \sum_{k=0}^3 R^{-4+k} |\nabla^k u|, \quad \text{in } B_R,$$

where N and R are positive numbers. Let $\beta > \sqrt{\frac{\mu^* \nu^*}{\mu_* \nu_*}} - 1$. There exist positive constants $s_1 \in (0, 1)$ and $C \geq 1$, C and s_1 only depending on $\lambda, \Lambda, \Lambda_1$ and N such that, for every $\rho_1 \in (0, s_1 R)$ and for every r, ρ satisfying $r < \rho < \frac{\rho_1 \lambda^2}{2}$,

$$(5.36) \quad \sum_{k=0}^3 \rho^{2k} \int_{B_\rho} |\nabla^k u|^2 \leq C \max \left\{ 1, \left(\frac{\rho}{R} \right)^{-(5\beta-2)} \right\} e^{C \left((\lambda^{-1} \rho)^{-\beta} - \left(\frac{\rho_1 \lambda}{2} \right)^{-\beta} \right) R^\beta} \cdot \left(\left(\frac{r}{R} \right)^{5\beta-2} \sum_{k=0}^3 r^{2k} \int_{B_r} |\nabla^k u|^2 \right)^{\vartheta_0} \cdot \left(\left(\frac{\rho_1}{R} \right)^{5\beta-2} \sum_{k=0}^3 \rho_1^{2k} \int_{B_{\rho_1}} |\nabla^k u|^2 \right)^{1-\vartheta_0},$$

where

$$(5.37) \quad \vartheta_0 = \frac{(\lambda^{-1} \rho)^{-\beta} - \left(\frac{\lambda \rho_1}{2} \right)^{-\beta}}{\left(\frac{\lambda r}{2} \right)^{-\beta} - \left(\frac{\lambda \rho_1}{2} \right)^{-\beta}}.$$

Proof. First we observe that, denoting $\tilde{g}_k^{-1}(x) = g_k^{-1}(Rx)$, $\tilde{L}_k = \tilde{g}_k^{ij}(x) \partial_{ij}^2$, $k = 1, 2$, $\tilde{\mathcal{L}} = \tilde{L}_2 \tilde{L}_1$, $\tilde{u}(x) = u(Rx)$, $x \in B_1$, inequality (5.35) implies

$$(5.38) \quad |\tilde{\mathcal{L}}\tilde{u}| \leq N \sum_{k=0}^3 |\nabla^k \tilde{u}|, \quad \text{in } B_1.$$

For simplicity of notation we shall omit the symbol \sim . Let us introduce the following notation

$$(5.39) \quad J(\rho) = \sum_{k=0}^3 \rho^{2k} \int_{B_\rho^{\sigma_0}} |\nabla^k u|^2,$$

where, we recall, $B_\rho^{\sigma_0} = \{x \in \mathbb{R}^n \mid \sigma_0(x) < \rho\}$ and σ_0 has been defined in Theorem 5.2. Notice that (5.13) gives $B_{\lambda r} \subset B_r^{\sigma_0} \subset B_{\frac{r}{\lambda}}$, for every $r > 0$. In particular inequality (5.38) is satisfied in $B_\lambda^{\sigma_0}$. Denote by $R_1 = \min\{r_1, \lambda\}$, where r_1 has been introduced in Theorem 5.2. Let $\rho_1 \in (0, R_1]$ and $r \in$

$(0, \frac{\rho_1}{2})$. Let $\eta \in C_0^4(\mathbb{R})$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $(r, \frac{\rho_1}{2})$, $\eta \equiv 0$ in $(0, \frac{r}{2}) \cup (\rho_1, R_1)$, $\left| \frac{d^k}{dt^k} \eta \right| \leq \frac{C}{r^k}$ in $[\frac{r}{2}, r]$, $\left| \frac{d^k}{dt^k} \eta \right| \leq \frac{C}{\rho_1^k}$ in $[\frac{\rho_1}{2}, \rho_1]$ for $k = 0, 1, \dots, 4$, where C is an absolute constant. In addition, let $\xi(x) = \eta(\sigma_0(x))$. By a standard density theorem, inequality (5.15) holds for the function $\xi(x)u(x)$.

Denote

$$(5.40) \quad h_\tau(t) = t^{5\beta-2} e^{\frac{2\tau}{t^\beta}}, \quad t \in (0, 1).$$

By standard calculations, it is simple to derive that there exist $\bar{\tau}_1 \geq \bar{\tau}$, C , $s_0 \in (0, R_1)$, only depending on λ , Λ , Λ_1 , β and N , such that if $\rho_1 \leq s_0$, $r < \rho < \frac{\rho_1}{2}$ and $\tau \geq \bar{\tau}_1$ then

$$(5.41) \quad h_\tau(\rho)J(\rho) \leq Ch_\tau\left(\frac{r}{2}\right)J(r) + Ch_\tau\left(\frac{\rho_1}{2}\right)J(\rho_1).$$

Hence

$$(5.42) \quad J(\rho) \leq C \left(\left(\frac{r/2}{\rho} \right)^{5\beta-2} e^{2\tau \left(-\frac{1}{\rho^\beta} + \frac{1}{(r/2)^\beta} \right)} J(r) + \left(\frac{\rho_1/2}{\rho} \right)^{5\beta-2} e^{2\tau \left(-\frac{1}{\rho^\beta} + \frac{1}{(\rho_1/2)^\beta} \right)} J(\rho_1) \right),$$

for every $\tau \geq \bar{\tau}_1$.

Let us denote

$$(5.43) \quad \tilde{\vartheta}_0 = \frac{\rho^{-\beta} - \left(\frac{\rho_1}{2}\right)^{-\beta}}{\left(\frac{r}{2}\right)^{-\beta} - \left(\frac{\rho_1}{2}\right)^{-\beta}},$$

$$(5.44) \quad \alpha_0 = \frac{1}{2} \frac{\log \left(\left(\frac{\rho_1}{r} \right)^{5\beta-2} \frac{J(\rho_1)}{J(r)} \right)}{\left(\frac{r}{2}\right)^{-\beta} - \left(\frac{\rho_1}{2}\right)^{-\beta}}.$$

If $\alpha_0 \geq \bar{\tau}_1$ then we choose $\tau = \alpha_0$ in (5.42) obtaining

$$(5.45) \quad J(\rho) \leq \frac{C}{\rho^{5\beta-2}} (r^{5\beta-2} J(r))^{\vartheta_0} \left(\rho_1^{5\beta-2} J(\rho_1) \right)^{1-\vartheta_0},$$

where C only depends on λ , Λ , Λ_1 , N and β .

If $\alpha_0 < \bar{\tau}_1$ then we have trivially

$$(5.46) \quad J(\rho) \leq J(\rho_1) = (J(\rho_1))^{\vartheta_0} (J(\rho_1))^{1-\vartheta_0} \leq \frac{e^{2\bar{\tau}_1 \left(\rho^{-\beta} - \left(\frac{\rho_1}{2}\right)^{-\beta} \right)}}{\rho_1^{5\beta-2}} (r^{5\beta-2} J(r))^{\vartheta_0} \left(\rho_1^{5\beta-2} J(\rho_1) \right)^{1-\vartheta_0}.$$

By (5.45) and (5.46) and scaling the variables we get (5.36). \square

Corollary 5.4 (Unique continuation property). *Let \mathcal{L} be the same operator of Theorem 5.3 and let ν_* , ν^* , μ_* , μ^* be as defined in Theorem 5.2. Let us assume that $u \in H^4(B_R)$ satisfies the inequality*

$$(5.47) \quad |\mathcal{L}u| \leq N \sum_{k=0}^3 R^{-4+k} |\nabla^k u|, \quad \text{in } B_R,$$

where N and R are positive numbers.

Assume that

$$(5.48) \quad \int_{B_r} u^2 = O\left(e^{-\frac{C_0}{r^\kappa}}\right), \quad \text{as } r \rightarrow 0,$$

where $C_0 > 0$ and $\kappa > \sqrt{\frac{\mu^* \nu^*}{\mu_* \nu_*}} - 1$.

Then we have

$$(5.49) \quad u \equiv 0 \quad \text{in } B_R.$$

Proof. Let us fix $\rho_1 \in (0, s_1 R)$ and $\rho \in \left(r, \frac{\lambda^2}{2} \rho_1\right)$, where s_1 has been defined in Theorem 5.3. Let

$$(5.50) \quad \sqrt{\frac{\mu^* \nu^*}{\mu_* \nu_*}} - 1 < \beta < \kappa.$$

By (5.36) and by the interpolation inequality

$$(5.51) \quad \|u\|_{H^3(B_r)} \leq C \|u\|_{L^2(B_r)}^{\frac{1}{4}} \|u\|_{H^4(B_r)}^{\frac{3}{4}},$$

where $C > 0$ is an absolute constant, we have

$$(5.52) \quad \|u\|_{H^3(B_\rho)}^2 \leq C \left(\left(\frac{r}{R} \right)^{5\beta-2} \|u\|_{L^2(B_r)}^{\frac{1}{2}} \right)^{\vartheta_0},$$

where ϑ_0 is given by (5.37) and $C > 0$ only depends on λ , Λ , Λ_1 , N , β , ρ , ρ_1 , R and $\|u\|_{H^4(B_R)}$. By (5.48) and (5.50), passing to the limit as $r \rightarrow 0$ in (5.52), we obtain $u \equiv 0$ in B_ρ . By iteration the thesis follows. \square

6 Three sphere inequalities for the plate operator

In this section we specialize the results of Section 5, in particular we specialize the three sphere inequality proved in Theorem 5.3, for the plate equation

$$(6.1) \quad \mathcal{L}u := \partial_{ij}^2 (C_{ijkl} \partial_{kl}^2 u) = 0, \quad \text{in } B_R,$$

where $\{C_{ijkl}(x)\}_{i,j,k,l=1}^2$ is a fourth order tensor that satisfies the hypotheses (3.2), (3.7), (3.8) for $\Omega = B_R$ and the *dichotomy condition* in B_R .

In the following, without loss of generality, we assume $R = 1$.

In order to apply Theorem 5.2 we need to write the operator \mathcal{L} in the following form

$$(6.2) \quad \mathcal{L} = L_2 L_1 + \tilde{Q},$$

where L_1 and L_2 are second order operators which satisfy a uniform ellipticity condition and whose coefficients belong to $C^{1,1}(B_1)$ and \tilde{Q} is a third order operator with bounded coefficients. In the sequel (Lemma 6.1) we shall prove that (6.2) holds true under some additional assumptions on the tensor $\{C_{ijkl}(x)\}_{i,j,k,l=1}^2$.

Let us denote

$$(6.3) \quad p(x; \partial)u = \sum_{h=0}^4 a_{4-h}(x) \partial_1^h \partial_2^{4-h} u, \quad \text{for every } u \in H^4(B_1),$$

where the coefficients $a_i(x)$, $i = 0, \dots, 4$, have been defined in (3.9), (3.10).

By (3.9) we have

$$(6.4) \quad \mathcal{L}u = p(x; \partial)u + Qu, \quad \text{for every } u \in H^4(B_1),$$

where Q is a third order operator with bounded coefficients which satisfies the inequality

$$(6.5) \quad |Qu| \leq cM (|\nabla^3 u| + |\nabla^2 u|), \quad \text{for every } u \in H^4(B_1),$$

and c is an absolute constant. In addition we denote

$$(6.6) \quad p(x; \xi) = \sum_{h=0}^4 a_{4-h}(x) \xi_1^h \xi_2^{4-h}, \quad x \in \overline{B_1}, \quad \xi \in \mathbb{R}^2,$$

$$(6.7) \quad \tilde{p}(x; t) := p(x; (t, 1)) = \sum_{h=0}^4 a_{4-h}(x) t^h, \quad x \in \overline{B_1}, \quad t \in \mathbb{R}.$$

Notice that by (3.7) we have

$$(6.8) \quad p(x; \xi) \geq \gamma |\xi|^4, \quad x \in \overline{B_1}, \quad \xi \in \mathbb{R}^2,$$

$$(6.9) \quad \tilde{p}(x; t) \geq \gamma (t^2 + 1)^2, \quad x \in \overline{B_1}, \quad t \in \mathbb{R}.$$

Now, for any fixed $x \in \overline{B}_1$, let $z_k(x) = \alpha_k(x) + i\beta_k(x)$, $\overline{z}_k(x) = \alpha_k(x) - i\beta_k(x)$ ($k = 1, 2$) be the complex solutions to the algebraic equation $\tilde{p}(x; z) = 0$. Here, α_k and β_k are real-valued functions and $\beta_k(x) > 0$, $k = 1, 2$, for every $x \in \overline{B}_1$.

We have

$$(6.10) \quad p(x; \xi) = p_2(x; \xi)p_1(x; \xi), \quad \text{for every } x \in \overline{B}_1, \xi \in \mathbb{R}^2,$$

where

$$(6.11) \quad p_k(x; \xi) = g_k^{ij}(x)\xi_i\xi_j, \quad k = 1, 2, \quad x \in \overline{B}_1, \xi \in \mathbb{R}^2,$$

$$(6.12) \quad g_k^{11}(x) = \sqrt{a_0(x)}, \quad g_k^{12}(x) = g_k^{21}(x) = -\alpha_k(x)\sqrt{a_0(x)}, \\ g_k^{22}(x) = \sqrt{a_0(x)}(\alpha_k^2(x) + \beta_k^2(x)), \quad k = 1, 2, \quad x \in \overline{B}_1.$$

Since in the sequel we have to deal with some basic properties of polynomials, we recall such properties for what concerns the polynomial $\tilde{p}(x; z)$ and we refer the reader to [Wa, Chapter 5] for an extended treatment of the issue. For any fixed $x \in \overline{B}_1$ we denote by $\mathcal{D}(x)$ the absolute value of the discriminant of the polynomial $\tilde{p}(x; z)$, that is

$$(6.13) \quad \mathcal{D}(x) = a_0^6 ((z_1 - z_2)(z_1 - \overline{z}_1)(z_1 - \overline{z}_2)(z_2 - \overline{z}_1)(z_2 - \overline{z}_2)(\overline{z}_1 - \overline{z}_2))^2,$$

where $a_0 = a_0(x)$ and $z_k = z_k(x) = \alpha_k(x) + i\beta_k(x)$, $k = 1, 2$. An elementary calculation yields

$$(6.14) \quad \mathcal{D}(x) = 16a_0^6\beta_1^2\beta_2^2 [(\alpha_1 - \alpha_2)^2 + (\beta_1 + \beta_2)^2]^2 [(\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2]^2.$$

In terms of the coefficients $a_h = a_h(x)$, $h = 0, 1, \dots, 4$, it is also known that

$$(6.15) \quad \mathcal{D}(x) = \frac{1}{a_0} |\det S(x)|,$$

where $S(x)$ is the 7×7 matrix defined by (3.11).

Furthermore, let us denote by Ψ the map of \mathbb{R}^4 into \mathbb{R}^4 defined by $\Psi(t_1, t_2, w_1, w_2) = \{\Psi_k(t_1, t_2, w_1, w_2)\}_{k=1}^4$, where

$$(6.16) \quad \begin{cases} \Psi_1(t_1, t_2, w_1, w_2) = t_1 + t_2, \\ \Psi_2(t_1, t_2, w_1, w_2) = t_1^2 + t_2^2 + 4t_1t_2 + w_1 + w_2, \\ \Psi_3(t_1, t_2, w_1, w_2) = t_1(t_2^2 + w_2) + t_2(t_1^2 + w_1), \\ \Psi_4(t_1, t_2, w_1, w_2) = (t_1^2 + w_1)(t_2^2 + w_2). \end{cases}$$

Notice that

$$(6.17) \quad a_1 = -2a_0\Psi_1(\alpha_1, \alpha_2, \beta_1^2, \beta_2^2),$$

$$(6.18) \quad a_2 = a_0\Psi_2(\alpha_1, \alpha_2, \beta_1^2, \beta_2^2),$$

$$(6.19) \quad a_3 = -2a_0\Psi_3(\alpha_1, \alpha_2, \beta_1^2, \beta_2^2),$$

$$(6.20) \quad a_4 = a_0\Psi_4(\alpha_1, \alpha_2, \beta_1^2, \beta_2^2).$$

Let us denote by $\frac{\partial\Psi(t_1, t_2, w_1, w_2)}{\partial(t_1, t_2, w_1, w_2)}$ the jacobian matrix of Ψ and let $J(t_1, t_2, w_1, w_2)$ be its determinant. An elementary calculation shows that

$$(6.21) \quad J(t_1, t_2, w_1, w_2) = -[(t_1 - t_2)^4 + 2(w_1 + w_2)(t_1 - t_2)^2 + (w_1 - w_2)^2].$$

Let us denote

$$(6.22) \quad \gamma_1 = \min \left\{ \gamma, \frac{1}{16M}, 1 \right\}.$$

The following lemma holds.

Lemma 6.1. *Let $p_k(x; \xi)$, $k = 1, 2$, be defined by (6.11). The following facts hold:*

(a) *If (3.2) and (3.7) are satisfied, then*

$$(6.23) \quad \gamma_2|\xi|_2^2 \leq p_k(x; \xi) \leq \gamma_2^{-1}|\xi|_2^2, \quad \text{for every } x \in \overline{B}_1, \xi \in \mathbb{R}^2, \quad k = 1, 2,$$

where $\gamma_2 = 5^{-6}\gamma_1^{15}$.

(b) *If the dichotomy condition introduced in Definition 3.1 holds true in B_1 , then $g_k^{ij} \in C^{1,1}(\overline{B}_1)$, for $i, j, k = 1, 2$.*

More precisely, if (3.18a) holds true, then

$$(6.24) \quad \sum_{i,j,k=1}^2 \left(\|\nabla g_k^{ij}\|_{L^\infty(B_1)} \delta_1^{1/2} + \|\nabla^2 g_k^{ij}\|_{L^\infty(B_1)} \delta_1 \right) \leq C_1,$$

where $\delta_1 = \min_{\overline{B}_1} \mathcal{D}(x)$ and C_1 only depends on M and γ , whereas if (3.18b) holds true, then

$$(6.25) \quad \sum_{i,j,k=1}^2 \left(\|\nabla g_k^{ij}\|_{L^\infty(B_1)} + \|\nabla^2 g_k^{ij}\|_{L^\infty(B_1)} \right) \leq C_2,$$

where C_2 only depends on M and γ .

Proof. First we prove (a). Let $x, x \in \overline{B}_1$, be fixed. In the rest of the proof of (a) we shall omit, for brevity, the dependence on x .

By (6.8), (3.7), (6.22), we have

$$(6.26) \quad \gamma_1 |\xi|^4 \leq p(\xi) \leq \gamma_1^{-1} |\xi|^4, \quad \text{for every } \xi \in \mathbb{R}^2.$$

Now we observe that the following inequalities hold true

$$(6.27) \quad |\alpha_1 + \alpha_2| \leq \gamma_1^{-2},$$

$$(6.28) \quad |\alpha_1^2 + \beta_1^2 + \alpha_2^2 + \beta_2^2 + 4\alpha_1\alpha_2| \leq \gamma_1^{-2},$$

$$(6.29) \quad |\alpha_1(\alpha_2^2 + \beta_2^2) + \alpha_2(\alpha_1^2 + \beta_1^2)| \leq \gamma_1^{-2},$$

$$(6.30) \quad \gamma_1^2 \leq (\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2) \leq \gamma_1^{-2},$$

$$(6.31) \quad \gamma_1^2(1 + \alpha_1^2)^2 \leq \beta_1^2 [(\alpha_1 - \alpha_2)^2 + \beta_2^2] \leq \gamma_1^{-2}(1 + \alpha_1^2)^2,$$

$$(6.32) \quad \gamma_1^2(1 + \alpha_2^2)^2 \leq \beta_2^2 [(\alpha_1 - \alpha_2)^2 + \beta_1^2] \leq \gamma_1^{-2}(1 + \alpha_2^2)^2.$$

Indeed, by (6.26) we have

$$(6.33) \quad \gamma_1 \leq a_0 \leq \gamma_1^{-1}, \quad \gamma_1 \leq a_4 \leq \gamma_1^{-1}.$$

On the other hand, by (6.33) and using (6.17), (6.18), (6.19), (6.20) we obtain the inequalities (6.27), (6.28), (6.29), (6.30), respectively. Concerning (6.31), by using (6.26) for $\xi = (\alpha_1, 1)$ and taking into account (6.10), we have

$$(6.34) \quad \gamma_1(1 + \alpha_1^2)^2 \leq a_0\beta_1^2 [(\alpha_1 - \alpha_2)^2 + \beta_2^2] \leq \gamma_1^{-1}(1 + \alpha_1^2)^2.$$

Inequality (6.31) follows from the first of (6.33) and (6.34). Proceeding similarly for $\xi = (\alpha_2, 1)$ we obtain (6.32).

Now, denoting

$$(6.35) \quad \epsilon_0 = \frac{\gamma_1^3}{\sqrt{50}},$$

we are going to prove that the following inequalities hold

$$(6.36) \quad \beta_k > \epsilon_0, \quad k = 1, 2,$$

$$(6.37) \quad \beta_k \leq \frac{1}{\gamma_1 \epsilon_0}, \quad k = 1, 2,$$

$$(6.38) \quad |\alpha_k| \leq \frac{1}{\gamma_1 \epsilon_0}, \quad k = 1, 2.$$

In order to prove (6.36), it is enough to consider the case $k = 1$, as the case $k = 2$ can be proved by the same arguments. We proceed by contradiction and we assume that

$$(6.39) \quad \beta_1^2 \leq \epsilon_0^2.$$

By (6.39) and (6.31) we get

$$(6.40) \quad \frac{\gamma_1^2}{\epsilon_0^2} \leq (\alpha_1 - \alpha_2)^2 + \beta_2^2,$$

hence at least one of the following inequalities must hold

$$(6.41) \quad \frac{\gamma_1^2}{2\epsilon_0^2} \leq \beta_2^2,$$

$$(6.42) \quad \frac{\gamma_1^2}{2\epsilon_0^2} \leq (\alpha_1 - \alpha_2)^2.$$

If the inequality (6.41) holds, then by (6.30) we have

$$(6.43) \quad \alpha_1^2 \leq \alpha_1^2 + \beta_1^2 \leq \frac{\gamma_1^{-2}}{\alpha_2^2 + \beta_2^2} \leq \frac{\gamma_1^{-2}}{\beta_2^2} \leq 2\gamma_1^{-4}\epsilon_0^2,$$

hence

$$(6.44) \quad |\alpha_1| \leq \sqrt{2}\gamma_1^{-2}\epsilon_0,$$

and in turn inequalities (6.44), (6.27) imply

$$(6.45) \quad |\alpha_2| \leq (1 + \sqrt{2}\epsilon_0)\gamma_1^{-2}.$$

Therefore, by (6.28), (6.41), (6.44), (6.45), and recalling that $\gamma_1 \in (0, 1)$, we have

$$(6.46) \quad \frac{\gamma_1^2}{2\epsilon_0^2} \leq \beta_2^2 \leq \alpha_2^2 + \beta_2^2 + \alpha_1^2 + \beta_1^2 < 25\gamma_1^{-4},$$

hence we have $\epsilon_0 > \frac{\gamma_1^3}{\sqrt{30}}$, a contradiction. Hence, (6.41) cannot be true.

If (6.42) holds, then we have $|\alpha_1| + |\alpha_2| \geq |\alpha_1 - \alpha_2| \geq \frac{\gamma_1}{\sqrt{2\epsilon_0}}$. Therefore, at least one of the following inequalities holds

$$(6.47) \quad |\alpha_1| \geq \frac{\gamma_1}{2\sqrt{2\epsilon_0}}, \quad |\alpha_2| \geq \frac{\gamma_1}{2\sqrt{2\epsilon_0}}.$$

If the first of (6.47) holds, then by (6.27) we have $|\alpha_2| \geq |\alpha_1| - \gamma_1^{-2} \geq \frac{\gamma_1}{2\sqrt{2\epsilon_0}} - \gamma_1^{-2} \geq \frac{\gamma_1}{4\sqrt{2\epsilon_0}}$ and, analogously, if the second of (6.47) holds, then we have $|\alpha_1| \geq \frac{\gamma_1}{4\sqrt{2\epsilon_0}}$. Hence, if (6.42) holds, then we have

$$(6.48) \quad |\alpha_1| \geq \frac{\gamma_1}{4\sqrt{2\epsilon_0}}, \quad |\alpha_2| \geq \frac{\gamma_1}{4\sqrt{2\epsilon_0}}.$$

Inequalities (6.48) and (6.30) give

$$(6.49) \quad \frac{\gamma_1^2}{32\epsilon_0^2} \leq \alpha_1^2 \leq \alpha_1^2 + \beta_1^2 \leq \frac{\gamma_1^{-2}}{\alpha_2^2 + \beta_2^2} \leq \frac{\gamma_1^{-2}}{\alpha_2^2} \leq 32\gamma_1^{-4}\epsilon_0^2.$$

As a consequence of the above inequality we have $\frac{\gamma_1^3}{32} \leq \epsilon_0^2$, that contradicts (6.35). Therefore, (6.39) cannot be true and (6.36) is proved.

By (6.30) and (6.36) we easily obtain (6.37) and (6.38). Finally, by (6.36)–(6.38), we obtain easily an estimate from above and from below of the eigenvalues of the matrices $\{g_k^{ij}(x)\}_{i,j=1}^2$ from which the estimate (6.23) follows.

Now we prove the statement (b) of the lemma. By (6.21), (6.33), (6.36)–(6.38) we have

$$(6.50) \quad \gamma_3 \sqrt{\mathcal{D}(x)} \leq J(x) \leq \gamma_3^{-1} \sqrt{\mathcal{D}(x)}, \quad \text{for every } x \in \overline{B}_1,$$

where

$$(6.51) \quad J(x) = |J(\alpha_1(x), \alpha_2(x), \beta_1^2(x), \beta_2^2(x))|$$

and $\gamma_3 = 10^{-6}\gamma_1^{25}\gamma_0^{-3}$.

Assume that (3.18a) holds in B_1 . In order to prove that $g_k^{ij} \in C^{1,1}(\overline{B}_1)$ and to derive estimate (6.24), it is enough to apply the Inverse Mapping Theorem to the map Ψ . Indeed, by (6.16), the vector-valued function $\omega(x) = (\alpha_1(x), \alpha_2(x), \beta_1^2(x), \beta_2^2(x))$ satisfies the following equality

$$(6.52) \quad \Psi(\omega(x)) = d(x), \quad x \in \overline{B}_1,$$

where $d(x) = \left(-\frac{a_1(x)}{2a_0(x)}, \frac{a_2(x)}{a_0(x)}, -\frac{a_3(x)}{2a_0(x)}, \frac{a_4(x)}{a_0(x)}\right)$, hence by (3.8), (3.9), (3.10), (6.50), (6.51), (6.52) we obtain (6.24).

If (3.18b) holds true, then by (6.14) we have $\alpha_1(x) = \alpha_2(x)$ and $\beta_1(x) = \beta_2(x)$ for every $x \in \overline{B}_1$. Therefore, by (6.16)–(6.18) we have

$$(6.53) \quad \alpha_1(x) = \alpha_2(x) = -\frac{a_1(x)}{4a_0(x)}$$

and

$$(6.54) \quad \beta_1^2(x) = \beta_2^2(x) = \frac{a_2(x)}{2a_0(x)} - \frac{3a_1^2(x)}{16a_0^2(x)}.$$

By (3.8), (3.9), (3.10), (6.33), (6.36), (6.53) and (6.54) we get (6.25). \square

Theorem 6.2 (Three sphere inequality - first version). *Let us assume that $u \in H^4(B_R)$ is a solution to the equation*

$$(6.55) \quad \partial_{ij}^2(C_{ijkl}(x)\partial_{kl}^2 u) = 0, \quad \text{in } B_R,$$

where $\{C_{ijkl}(x)\}_{i,j,k,l=1}^2$ is a fourth order tensor whose entries belong to $C^{1,1}(\overline{B}_R)$. Assume that (3.2), (3.7), (3.8) and the dichotomy condition are satisfied in B_R . Let $\gamma_2 = 5^{-6}\gamma_1^{15}$ and $\beta = \frac{1}{\gamma_2^2} - 1$. There exist positive constants s_2 , $0 < s_2 < 1$, and C , $C > 1$, s_2 and C only depending on γ , M and on $\delta_1 = \min_{\overline{B}_R} \mathcal{D}$, such that, for every $\rho_1 \in (0, s_2 R)$ and every r, ρ satisfying $r < \rho < \frac{\rho_1 \gamma_2^2}{2}$, the following inequality holds

$$(6.56) \quad \sum_{k=0}^3 \rho^{2k} \int_{B_\rho} |\nabla^k u|^2 \leq C \exp \left(C \left((\gamma_2^{-1} \rho)^{-\beta} - (\gamma_2 \frac{\rho_1}{2})^{-\beta} \right) R^\beta \right) \cdot \left(\sum_{k=0}^3 r^{2k} \int_{B_r} |\nabla^k u|^2 \right)^{\theta_1} \left(\sum_{k=0}^3 \rho_1^{2k} \int_{B_{\rho_1}} |\nabla^k u|^2 \right)^{1-\theta_1},$$

where

$$(6.57) \quad \theta_1 = \frac{(\gamma_2^{-1} \rho)^{-\beta} - (\gamma_2 \frac{\rho_1}{2})^{-\beta}}{(\gamma_2 \frac{r}{2})^{-\beta} - (\gamma_2 \frac{\rho_1}{2})^{-\beta}}.$$

Proof. Let us define

$$(6.58) \quad \tilde{u}(y) = u(Ry), \quad \tilde{C}_{ijkl}(y) = C_{ijkl}(Ry), \quad y \in \overline{B}_1, \quad i, j, k, l = 1, 2.$$

Then, $\tilde{u} \in H^4(B_1)$ is a solution to the equation

$$(6.59) \quad \partial_{ij}^2(\tilde{C}_{ijkl}(y)\partial_{kl}^2 \tilde{u}) = 0, \quad \text{in } B_1.$$

Now, by Lemma 6.1 we have that

$$(6.60) \quad \mathcal{L} = L_2 L_1 \tilde{u} + Q \tilde{u},$$

where $L_k = p_k(y; \partial)$, $k = 1, 2$, and

$$(6.61) \quad p_k(y; \partial) = g_k^{ij} \partial_{ij}^2, \quad k = 1, 2.$$

Here, $\{g_k^{ij}\}_{i,j=1}^2$, $k = 1, 2$, satisfy (6.24) or (6.25) (the former whenever (3.18a) holds, the latter whenever (3.18b) holds),

$$(6.62) \quad \gamma_2 |\xi|^2 \leq g_k^{ij}(y) \xi_i \xi_j \leq \gamma_2^{-1} |\xi|^2, \quad x \in \overline{B}_1, \quad \xi \in \mathbb{R}^2,$$

and Q is a third order operator with bounded coefficients satisfying

$$(6.63) \quad |Q \tilde{u}| \leq cM (|\nabla^3 \tilde{u}| + |\nabla^2 \tilde{u}|),$$

where c is an absolute constant. Therefore, from (6.60)–(6.63) and Theorem 5.3, and coming back to the old variables, we obtain the three sphere inequality (6.56). \square

The following Poincaré-type inequality holds.

Proposition 6.3 (Poincaré inequality). *There exists a positive constant C only depending on n such that for every $u \in H^2(B_R, \mathbb{R}^n)$ and for every $r \in (0, R]$*

$$(6.64) \quad \int_{B_R} |\tilde{u}_r|^2 + R^2 \int_{B_R} |\nabla \tilde{u}_r|^2 \leq CR^4 \left(\frac{R}{r}\right)^n \int_{B_R} |\nabla^2 u|^2,$$

where

$$(6.65) \quad \tilde{u}_r(x) = u(x) - (u)_r - (\nabla u)_r \cdot x,$$

$$(6.66) \quad (u)_r = \frac{1}{|B_r|} \int_{B_r} u, \quad (\nabla u)_r = \frac{1}{|B_r|} \int_{B_r} \nabla u.$$

Proof. For a proof we refer to [A-M-Ro4, Example 4.3]. \square

Proposition 6.4 (Caccioppoli-type inequality). *Let us assume that $u \in H^4(B_R)$ is a solution to the equation*

$$(6.67) \quad \partial_{ij}^2 (C_{ijkl}(x) \partial_{kl}^2 u) = 0, \quad \text{in } B_R,$$

where $\{C_{ijkl}(x)\}_{i,j,k,l=1}^2$ is a fourth order tensor whose entries belong to $C^{1,1}(\overline{B}_R)$. Assume that (3.2)–(3.8) are satisfied. We have

$$(6.68) \quad \int_{B_{\frac{t}{2}}} |\nabla^3 u|^2 \leq C \int_{B_t} \sum_{k=0}^2 (t^{k-3} |\nabla^k u|)^2, \quad \text{for every } t \leq R,$$

where C is a positive constant only depending on γ and M .

Proof. The proof of (6.68) is essentially the same of the proof of [M-Ro-Ve1, Proposition 6.2]. Here, for the reader convenience, we give a sketch of the proof.

For every $t \in (0, R]$, let $\eta \in C_0^\infty(B_t)$ be such that $0 \leq \eta \leq 1$ in B_t , $\eta \equiv 1$ in $B_{\frac{t}{2}}$ and

$$(6.69) \quad \sum_{k=1}^3 t^k |\nabla^k \eta| \leq C, \quad \text{in } B_t,$$

where C is an absolute constant. Multiplying equation (6.67) by $\Delta(\eta^6 u)$ and integrating over B_t , we have

$$(6.70) \quad \int_{B_t} C_{ijkl} \partial_{kl}^2 u \partial_{ij}^2 \Delta(\eta^6 u) = 0$$

and, integrating by parts,

$$(6.71) \quad \int_{B_t} \{C_{ijkl} \partial_{kl}^2 \partial_s u \partial_{ij}^2 \partial_s(\eta^6 u) + \partial_s(C_{ijkl}) \partial_{kl}^2 u \partial_{ij}^2 \partial_s(\eta^6 u)\} = 0.$$

By (3.8), (6.69), (6.71) and taking into account that $t \leq R$ we have

$$(6.72) \quad \int_{B_t} \eta^6 C_{ijkl} \partial_{kl}^2 \partial_s u \partial_{ij}^2 \partial_s u = F[u],$$

where F satisfies the inequality

$$(6.73) \quad |F[u]| \leq CM \int_{B_t} \left(\sum_{k=0}^2 t^{k-3} |\nabla^k u| \right)^2 + CM \int_{B_t} |\nabla^3 u| \eta^3 \left(\sum_{k=0}^2 t^{k-3} |\nabla^k u| \right),$$

where C is an absolute constant. By (6.72), (6.73), (3.7) and Cauchy inequality ($2ab \leq \epsilon a^2 + \frac{1}{\epsilon} b^2$, for $\epsilon > 0$) we have

$$(6.74) \quad \gamma \int_{B_t} \eta^6 |\nabla^3 u|^2 \leq CM^2 \int_{B_t} \left(\sum_{k=0}^2 t^{k-3} |\nabla^k u| \right)^2.$$

Inequality (6.68) follows immediately by (6.74). \square

Theorem 6.5 (Three sphere inequality - second version). *Let $u \in H^4(B_R)$ be a solution to the equation*

$$(6.75) \quad \partial_{ij}^2(C_{ijkl}(x) \partial_{kl}^2 u) = 0, \quad \text{in } B_R,$$

where $\{C_{ijkl}(x)\}_{i,j,k,l=1}^2$ is a fourth order tensor whose entries belong to $C^{1,1}(\overline{B}_R)$. Assume that (3.2), (3.7), (3.8) and the dichotomy condition are satisfied in B_R . Let $\gamma_2 = 5^{-6}\gamma_1^{15}$ and $\beta = \frac{1}{\gamma_2^2} - 1$. There exist positive constants s , $0 < s < 1$, and C , $C \geq 1$, s and C only depending on γ , M and on $\delta_1 = \min_{\overline{B}_R} \mathcal{D}$, such that, for every $\rho_1 \in (0, sR)$ and every r, ρ satisfying $r < \rho < \frac{\rho_1 \gamma_2^2}{2}$, the following inequality holds

$$(6.76) \quad \rho^4 \int_{B_\rho} |\nabla^2 u|^2 \leq C \exp \left(C \left((\gamma_2^{-1} \rho)^{-\beta} - (\gamma_2 \frac{\rho_1}{2})^{-\beta} \right) R^\beta \right) \cdot \left(r^4 \int_{B_{2r}} |\nabla^2 u|^2 \right)^{\theta_1} \left(\frac{\rho_1^6}{r^2} \int_{B_{2\rho_1}} |\nabla^2 u|^2 \right)^{1-\theta_1},$$

where

$$(6.77) \quad \theta_1 = \frac{(\gamma_2^{-1} \rho)^{-\beta} - (\gamma_2 \frac{\rho_1}{2})^{-\beta}}{(\gamma_2 \frac{r}{2})^{-\beta} - (\gamma_2 \frac{\rho_1}{2})^{-\beta}}.$$

Proof. Let $a \in \mathbb{R}$, $\omega \in \mathbb{R}^2$ to be chosen later on. Since u is a solution to (6.75), also $v = u - a - \omega \cdot x$ is a solution to (6.75). By (6.56) we have

$$(6.78) \quad \rho^4 \int_{B_\rho} |\nabla^2 v|^2 \leq K (H_v(r))^{\theta_1} (H_v(\rho_1))^{1-\theta_1},$$

where

$$(6.79) \quad K = C \exp \left(C \left((\gamma_2^{-1} \rho)^{-\beta} - (\gamma_2 \frac{\rho_1}{2})^{-\beta} \right) R^\beta \right)$$

and

$$(6.80) \quad H_v(t) = \sum_{k=0}^3 t^{2k} \int_{B_t} |\nabla^k v|^2, \quad t \in (0, R).$$

By Proposition 6.4 we have

$$(6.81) \quad H_v(r) = C \sum_{k=0}^2 r^{2k} \int_{B_{2r}} |\nabla^k v|^2,$$

where C only depends on M and γ . Now, we choose

$$(6.82) \quad a = \frac{1}{|B_{2r}|} \int_{B_{2r}} u, \quad \omega = \frac{1}{|B_{2r}|} \int_{B_{2r}} \nabla u.$$

By Proposition 6.3 and from (6.81) we have

$$(6.83) \quad H_v(r) \leq Cr^4 \int_{B_{2r}} |\nabla^2 u|^2,$$

where C only depends on M and γ .

Similarly, by applying Propositions 6.3 and 6.4 we obtain

$$(6.84) \quad H_v(\rho_1) \leq C\rho_1^4 \left(\frac{\rho_1}{r}\right)^2 \int_{B_{2\rho_1}} |\nabla^2 u|^2,$$

where C only depends on γ and M . From (6.78), (6.81), (6.83), inequality (6.76) follows. \square

Theorem 6.6 (Three sphere inequality - third version). *Let $u \in H^4(B_R)$ be a solution to the equation*

$$(6.85) \quad \partial_{ij}^2(C_{ijkl}(x)\partial_{kl}^2 u) = 0, \quad \text{in } B_R,$$

where $\{C_{ijkl}(x)\}_{i,j,k,l=1}^2$ is a fourth order tensor whose entries belong to $C^{1,1}(\overline{B}_R)$. Assume that (3.2), (3.7), (3.8) and the dichotomy condition are satisfied in B_R . Let $\gamma_2 = 5^{-6}\gamma_1^{15}$ and $\beta = \frac{1}{\gamma_2} - 1$. There exist positive constants s , $0 < s < 1$, and C , $C \geq 1$, s and C only depending on γ , M and on $\delta_1 = \min_{\overline{B}_R} \mathcal{D}$, such that, for every $\rho_1 \in (0, sR)$ and every r, ρ satisfying $r < \rho < \frac{\rho_1 \gamma_2^2}{2}$, the following inequality holds

$$(6.86) \quad \int_{B_\rho} u^2 \leq C \exp \left(C((\gamma_2^{-1}\rho)^{-\beta} - (\gamma_2 \frac{\rho_1}{2})^{-\beta}) R^\beta \right) \cdot \left(\int_{B_r} u^2 \right)^\theta \left(\sum_{k=0}^4 \rho_1^{2k} \int_{B_{\rho_1}} |\nabla^k u|^2 \right)^{1-\theta},$$

where $\theta = \frac{\theta_1}{4}$, with θ_1 given by (6.57)

Proof. It follows immediately from (6.56) and by the interpolation inequality

$$\|u\|_{H^3(B_r)} \leq C \|u\|_{L^2(B_r)}^{\frac{1}{4}} \|u\|_{H^4(B_r)}^{\frac{3}{4}},$$

where C is an absolute constant and the norms are normalized according to the convention made in Section 3. \square

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